

# On Classifying Continuous Constraint Satisfaction Problems

Received Mar 7, 2022  
Revised Nov 3, 2023  
Accepted Feb 2, 2024  
Published Apr 16, 2024

**Key words and phrases**  
Constraint Satisfaction Problems,  
Existential Theory of the Reals,  
Computational Complexity,  
Classification

Tillmann Miltzow<sup>a</sup> ✉ 

Reinier F. Schmiermann<sup>b</sup> ✉ 

<sup>a</sup> Computer Science Department,  
Utrecht University

<sup>b</sup> Mathematics Department,  
Utrecht University

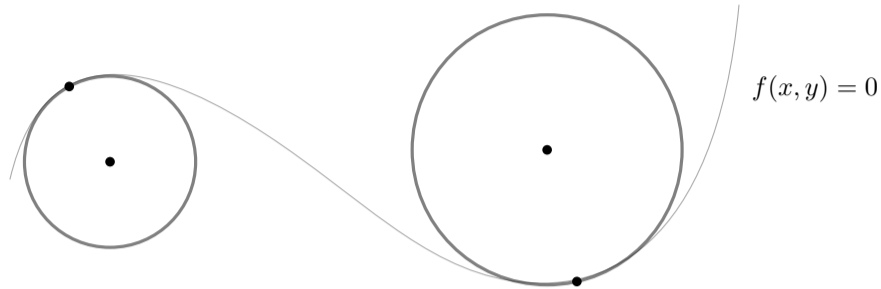
**ABSTRACT.** A continuous constraint satisfaction problem (CCSP) is a constraint satisfaction problem (CSP) with the real numbers as domain. We engage in a systematic study to classify CCSPs that are complete for the Existential Theory of the Reals, i.e.,  $\exists\mathbb{R}$ -complete. To define this class, we first consider the problem ETR, which also stands for Existential Theory of the Reals. In an instance of this problem we are given a sentence of the form  $\exists x_1, \dots, x_n \in \mathbb{R} : \Phi(x_1, \dots, x_n)$ , where  $\Phi$  is a well-formed quantifier-free formula consisting of the symbols  $\{0, 1, x_1, \dots, x_n, +, \cdot, \geq, >, \wedge, \vee, \neg\}$ . The goal is to check whether this sentence is true. Now the class  $\exists\mathbb{R}$  is the family of all problems that admit a polynomial-time many-one reduction to ETR. It is known that  $\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}$ .

We restrict our attention on CCSPs with addition constraints ( $x + y = z$ ) and which satisfy another mild technical condition. Previously, it was shown that multiplication constraints ( $x \cdot y = z$ ), squaring constraints ( $x^2 = y$ ), or inversion constraints ( $x \cdot y = 1$ ) are sufficient to establish  $\exists\mathbb{R}$ -completeness. We extend this in the strongest possible sense for equality constraints as follows. We show that CCSPs (with addition constraints and that satisfy another mild technical condition) that have *any* one well-behaved curved equality constraint ( $f(x, y) = 0$ ) are  $\exists\mathbb{R}$ -complete. We further extend our results to inequality constraints. We show that together *any* well-behaved convexly curved and *any* well-behaved concavely curved inequality constraint ( $f(x, y) \geq 0$  and  $g(x, y) \geq 0$ ) imply  $\exists\mathbb{R}$ -completeness on the class of such CCSPs.

Here, we call a function  $f : U \rightarrow \mathbb{R}$  well-behaved if it is a  $C^3$ -function,  $f(0, 0) = 0$ , all its first and second partial derivatives  $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$  are rational in  $(0, 0)$ ,  $f_x(0, 0) \neq 0$  or  $f_y(0, 0) \neq 0$ ,

A preliminary version of this article appeared at FOCS 2021. We thank Eunjung Kim and Saoirse Freeney for their helpful comments. We thank Mikkel Abrahamsen for letting us use parts of [4]. The first author is generously supported by the NWO Veni grant 016.Veni.192.250. The second author is generously supported by the NWO Vidi grant VI.Vidi.192.012. We would like to thank anonymous reviewers for their incredibly valuable feedback.

and another mild technical constraint. Furthermore, we call  $f$  curved if the curvature of the curve given by  $f(x, y) = 0$  is nonzero at the origin. In this case we call  $f$  either convexly curved if the curvature is negative, or concavely curved if it is positive.



**Figure 1.** The set defined by  $f(x, y) = 0$  is well-behaved and at some positions convexly curved and at others concavely curved, indicated by the two circles.

## 1. Introduction

In geometric packing, we are given a set of two-dimensional pieces, a container and a set of motions. The aim is to move the pieces into the container without overlap, and while respecting the given motions. Recently, Abrahamsen, Miltzow and Seiferth showed that many geometric packing variants are  $\exists\mathbb{R}$ -complete (FOCS 2020) [5]. Despite the fact that the first arXiv version is roughly 100 pages long, the high-level approach follows the same principle as many other hardness reductions. First, they showed that a technical intermediate problem is hard and then they reduced from this technical problem. In their work, ETR-INV, a specific continuous constraint satisfaction problem, serves as this intermediate  $\exists\mathbb{R}$ -complete problem. A complete definition of continuous constraint satisfaction problems is provided in the subsequent section. Specifically, ETR-INV contains essentially only addition constraints ( $x + y = z$ ) and inversion constraints ( $x \cdot y = 1$ ). In the second step, they showed how to encode addition and inversion using geometric objects. This enabled them to show in a unified framework that various geometric packing problems are  $\exists\mathbb{R}$ -complete.

The inversion constraint is particularly handy as it was shown in various other works that it is particularly easy to encode geometrically [38, 25, 26, 2]. Curiously, Abrahamsen, Miltzow and Seiferth left arguably the most interesting case of packing convex polygonal objects into a square container open. The missing puzzle piece seemed to be a gadget to encode the inversion constraint for this case.

We take an alternative approach and engage in a systematic study of continuous constraint satisfaction problems in their own respect. The aim is to fully classify all continuous constraint satisfaction problems by their computational complexity. Polynomial time, NP-complete, and  $\exists\mathbb{R}$ -complete are some apparent complexities, but as we will see, they may not be the only ones that are relevant, see Section 1.4. Our first application shows that packing convex polygons into

a square under rigid motions is  $\exists\mathbb{R}$ -complete. It arises as a combination of a small adaption of the framework by [5] and our structural results. As a result the paper by Abrahamsen, Miltzow and Seiferth considerably shortens to about 70 pages.

**REMARK 1.1.** Although the  $\exists\mathbb{R}$ -completeness of packing convex polygons into a square container under rigid motions was first pointed out in the conference version of this paper, the proof is more readable in the context of the paper by Abrahamsen, Miltzow and Seiferth [5]. The arxiv version of their paper incorporated the results presented in the conference version of this paper. In turn, some of the technical results of their paper are incorporated in this paper as Section 2 for the sake of readability and completeness.

We give a short introduction to constraint satisfaction problems and the complexity class  $\exists\mathbb{R}$ .

## 1.1 Constraint Satisfaction Problems

Constraint satisfaction problems (CSPs) are a wide class of computational decision problems. In order to give a formal definition, we first introduce several other terms.

**DEFINITION 1.2 (Signature).** A *signature* is a set of symbols together with *arities*  $\ell \in \mathbb{N}$ . Each symbol has exactly one arity attached to it.

Often the signature distinguishes between function symbols and relation symbols. We will only use relation symbols. We will only use signatures of finite size, to avoid dealing with issues of description complexity. For finite signatures, we can simply assume that each symbol has constant description complexity.

**DEFINITION 1.3 (Structure).** A structure consists of a set  $U$ , called the *domain*, a signature  $\tau$  and an *interpretation* of each symbol. If  $\alpha \in \tau$  is a symbol of arity  $\ell$ , then the interpretation is a set  $\alpha \subseteq U^\ell$ .

In the literature, the term *template* is also used as a synonym for structure. To make this more tangible, consider the following example. We define the domain  $U = \{0, 1\}$ , the symbol  $+_2$  of arity 3 and the symbol  $\mathbf{1}$  of arity 1. We interpret  $+_2$  as  $\{(x, y, z) \in U^3 \mid x + y \equiv z \pmod{2}\}$ , and  $\mathbf{1}$  as  $\{x \in U \mid x = 1\}$ . This defines a structure  $S_1 = \langle U, +_2, \mathbf{1} \rangle$ . Note that it is common to use a symbol and its interpretation interchangeably. Specifically, many symbols are used in the literature with their common interpretation, e.g.,  $\leq$  is interpreted as  $\{(x, y) \in U \mid x \leq y\}$  and  $+$  is interpreted as  $\{(x, y, z) \in U^3 \mid x + y = z\}$ . We refer to the symbols and interpretations of a structure merely as *constraints*. We will usually denote these constraints by the equation that they enforce. For example, we write  $x^2 = y$  for the constraint  $c = \{(x, y) \in U^2 \mid x^2 = y\}$ .

**DEFINITION 1.4 (Constraint satisfaction problem).** Given a structure  $S = (U, \tau)$  we define a *constraint formula*  $\Phi := \Phi(x_1, \dots, x_n)$  to be a conjunction  $c_1 \wedge \dots \wedge c_m$  for  $m \geq 0$ , where each  $c_i$  is of the form  $c(y_1, \dots, y_\ell)$  for some  $c \in \tau$  and variables  $y_1, \dots, y_\ell \in \{x_1, \dots, x_n\}$ . We also define  $V(\Phi) \subseteq U^n$  as  $V(\Phi) := \{\mathbf{x} \in U^n \mid \Phi(\mathbf{x})\}$ . In the *constraint satisfaction problem (CSP)* with structure  $S$ , we are given a constraint formula  $\Phi$ , and are asked whether  $V(\Phi) \neq \emptyset$ .

Consider the constraint formula  $\Phi = (x_1 + x_2 \equiv x_4 \pmod{2}) \wedge (x_2 + x_3 \equiv x_4 \pmod{2}) \wedge (x_2 = 1)$ . This gives an instance of a CSP with structure  $S_1$  as above. Note that  $(0, 1, 0, 1) \in V(\Phi)$ . It can be interesting whether the CSP with structure  $S_1$  is polynomial time solvable.

In this paper, we restrict ourselves to the reals as domain, i.e.,  $U = \mathbb{R}$  and denote them as *continuous constraint satisfaction problems (CCSPs)*.

We are mainly interested in CCSPs where the constraints are semi-algebraic over the integers (see Section 1.2 for a formal definition). There are some constraints that are not semi-algebraic, in other words, not computable on the real RAM [29]. For example, constraints involving  $\sin$ ,  $\cos$ ,  $\exp$ ,  $\log$  or testing if a number is an integer. We do not want to forbid those types of constraints in the general definition, as it might be interesting to study some of them. Some of our hardness results actually apply to non-computable functions. There are constraints that limit us to finite domains. For example,  $x(x-1) = 0$ . Although they are not truly continuous, they are indeed semi-algebraic and thus we have to deal with them as well in the general definition. For our results, we use both discrete constraints, e.g.  $x = 1$ , and truly continuous constraints, e.g.  $f(x) = y$  with  $f$  three times differentiable.

Constraint satisfaction problems have a long history in algorithmic studies [54, 19, 18, 63, 40, 27]. There are two application-driven motivations to study them. On the one hand, it is possible to easily encode many fundamental algorithmic problems directly as a CSP. Then, given an efficient algorithm for those types of CSPs, we have immediately also solved those other algorithmic problems. On the other hand, if we can encode CSPs into algorithmic problems, then any hardness result for the CSP immediately carries over to the algorithmic problem. Next to an application-driven motivation, it is fair to say that they deserve a study in their own right as fundamental mathematical objects. CSPs form a very versatile language and often allow for a complete classification by their computational complexity. Specifically, the *dichotomy conjecture* states that every class of CSP with a finite domain is either NP-complete or polynomial-time solvable. Schaefer showed the conjecture for domains of size two [54]. Recently, Bulatov and Zhuk could confirm the conjecture independently [18, 63] for any finite domain. Note that one can also try to find a classification from the parameterized complexity perspective [40] or the approximative counting perspective [27].

In this paper, we focus on CSPs with  $\mathbb{R}$  as domain and we are interested in the class of CSPs that are  $\exists\mathbb{R}$ -complete. We want to point out that there is also a large body of research that deals with infinite domains [64, 60, 13, 15, 34]. Most relevant for us is the work by Bodirsky, Jonsson and von Oertzen [14], who also studied CSPs over the reals and showed that a host of

them are NP-hard to decide. Specifically, they defined a subset  $S$  of  $\mathbb{R}^n$  as *essentially convex* if for all  $a, b \in S$ , the straight line segment intersects the complement  $\bar{S}$  of  $S$  in finitely many points. They show that CSPs that contain  $x = 1$ ,  $x \leq y$ ,  $x + y = z$ , and at least one constraint that is *not* essentially convex are NP-hard. However, their techniques do not imply  $\exists\mathbb{R}$ -hardness. See also [16] for an overview of results for the real domain.

## 1.2 Existential Theory of the Reals

The class of the existential theory of the reals  $\exists\mathbb{R}$  (pronounced as ‘ER’) is a complexity class that has gained a lot of interest, especially within the computational geometry community. To define this class, we first consider the problem ETR, which also stands for Existential Theory of the Reals. In an instance of this problem, we are given a sentence of the form

$$\exists x_1, \dots, x_n \in \mathbb{R} : \Phi(x_1, \dots, x_n),$$

where  $\Phi$  is a well-formed quantifier-free formula consisting of the symbols  $\{0, 1, x_1, \dots, x_n, +, \cdot, \geq, >, \wedge, \vee, \neg\}$ , the goal is to check whether this sentence is true. We will refer to the formula  $\Phi$  which might appear in an ETR-instance as an ETR-formula. As an example of an ETR-instance, we could take  $\Phi = (x \cdot y^2 + x \geq 0) \wedge \neg(y < 2x)$ . The goal of this instance would be to determine whether there exist real numbers  $x$  and  $y$  satisfying this formula. Now the class  $\exists\mathbb{R}$  is the family of all problems that admit a polynomial-time many-one reduction to ETR. With the notation above, it can be shown that the CSP of the structure  $\mathbf{R} = \langle \mathbb{R}, \cdot, +, \mathbf{1} \rangle$  is  $\exists\mathbb{R}$ -complete [41], see also Lemma 2.7.

It is known that

$$\text{NP} \subseteq \exists\mathbb{R} \subseteq \text{PSPACE}.$$

The first inclusion follows from the definition of  $\exists\mathbb{R}$  as follows. Given any Boolean satisfiability formula, we can replace each positive occurrence of a variable  $x$  by  $x = 1$ . For example  $(x \vee \neg y) \wedge (\neg x \vee z)$  becomes  $(x = 1 \vee \neg(y = 1)) \wedge (\neg(x = 1) \vee z = 1)$ .

Showing the second inclusion was first done by Canny in his seminal paper [20]. The reason that  $\exists\mathbb{R}$  is an important complexity class is that a number of common problems in computational geometry, game theory, machine learning, and other areas have been shown to be complete for this class.

We use  $|\Phi|$  to denote the length of  $\Phi$ , that is, the number of bits necessary to write down  $\Phi$ .

We want to point out that there are some subtleties in the definition of the formula length. Naively, to encode a natural number  $n$  requires  $\Theta(n)$  bits, i.e.,  $n = 1 + 1 + \dots + 1$ . However, it is possible to encode it in  $O(\log n)$  bits, using Horner’s rule applied to the binary expansion of  $n$ . For example,  $27 = 1 + 2(1 + 2(0 + 2(1 + 2))) = 1 + (1 + 1)(1 + (1 + 1)(0 + (1 + 1)(1 + (1 + 1))))$ . Furthermore, we want to emphasize that the reductions used for defining  $\exists\mathbb{R}$  are performed in

the word RAM model (or equivalently on a Turing machine), and not on a real Random Access Machine (real RAM) or in the Blum-Shub-Smale model.

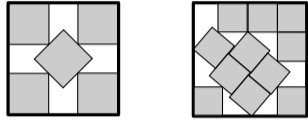
The definition of a formula naturally leads to the definition of semi-algebraic sets. We say a set  $S \subseteq \mathbb{R}^n$  is *semi-algebraic*, if there exists a formula  $\varphi$  such that  $S = \{x \in \mathbb{R}^n \mid \varphi(x)\}$ . Consequently, the (bit)-complexity of a semi-algebraic set is the shortest length of any formula defining the set. Note that our definition of a semi-algebraic set is more common in a computer science context [41]. In the context of algebraic geometry, semi-algebraic sets would usually allow polynomials with real coefficients. For example, consider the set  $S = \{x \in \mathbb{R} \mid x - e = 0\}$ , containing Euler number  $e$ . Note that  $S$  is typically semi-algebraic for an algebraic geometer [7], but typically not for a computer scientist [41]. Given a point  $x \in \mathbb{R}^n$  and a semi-algebraic set  $S \subseteq \mathbb{R}^n$ , we can decide on the real RAM if  $x \in S$ . (We refer the reader to the work by Erickson, Hoog, and Miltzow for a detailed definition of the real RAM and decidability [29].) This is easy to see as we only need to evaluate the defining formula of  $S$ . Interestingly the reverse direction also holds. If we can decide  $x \in S$  for any  $x \in \mathbb{R}^n$  then  $S$  needs to be semi-algebraic [29].

**Scope.** The main reason that the complexity class  $\exists\mathbb{R}$  gained traction in recent years is the increasing number of important algorithmic problems that are  $\exists\mathbb{R}$ -complete. Marcus Schaefer established the current name and pointed out first that several known NP-hardness reductions actually imply  $\exists\mathbb{R}$ -completeness [50]. Note that some important reductions that establish  $\exists\mathbb{R}$ -completeness were done before the class was named.

Problems that have a continuous solution space and non-linear relation between partial solutions are natural candidates to be  $\exists\mathbb{R}$ -complete. Early examples are related to the recognition of geometric structures: points in the plane [43, 57], geometric linkages [51], segment graphs [37, 41], unit disk graphs [42, 35], ray intersection graphs [21], and point visibility graphs [22]. In general, the complexity class is more established in the graph drawing community [38, 25, 49, 28]. Yet, it is also relevant for studying polytopes [48, 26]. There is a series of papers related to Nash-Equilibria [9, 52, 30, 12, 11]. Another line of research studies matrix factorization problems [23, 55, 56, 53]. Other  $\exists\mathbb{R}$ -complete problems are the Art Gallery Problem [2, 58], Covering polygons with convex polygons [1], and training neural networks [3, 10, 62].

**Practical Implications.** At first glance, the significance of  $\exists\mathbb{R}$ -completeness might not be immediately apparent, particularly given that most of these problems are already known to be NP-hard. The significance has different aspects. One reason is that we are intrinsically interested in establishing the true complexity of important algorithmic problems. Furthermore,  $\exists\mathbb{R}$ -completeness helps us to understand better the difficulties encountered when designing algorithms for those types of problems. While we have a myriad of techniques for NP-complete problems, most of these techniques are of limited use when we consider  $\exists\mathbb{R}$ -complete problems. The reason is that  $\exists\mathbb{R}$ -complete problems have an infinite set of possible solutions that are

intertwined in a sophisticated way. Many researchers have hoped to discretize the solution space, but success was limited [32, 41]. The complexity class  $\exists\mathbb{R}$  connects all of those different problems and tells us that we can either discretize all of them or none of them. To illustrate our lack of sufficient worst-case methods, note that we do not know the smallest square container to pack eleven unit squares, see Figure 2.



**Figure 2.** Left: Five unit squares into a minimum square container. Right: This is the best known packing of eleven unit squares into a square container [31].

**Technique.** In order to show  $\exists\mathbb{R}$ -hardness, usually two steps are involved. The first step is a reduction to a technical variant of ETR. The second step is a reduction from that variant to the problem at hand. Those ETR variants are typically CCSPs with only very limited types of constraints. It is common to have an addition constraint ( $x + y = z$ ), and a non-linear constraint, like one of the following:

$$z = x \cdot y, \quad z = x^2, \quad 1 = x \cdot y.$$

To find the right non-linear constraint is crucial for the second step, as it is often very difficult to encode non-linear constraints in geometric problems. Previous proof techniques relied on expressing multiplication indirectly using other operations. To be precise, we say that a constraint  $c$  of arity  $\ell$  has a *primitive positive definition* in structure  $S$ , if there is a constraint formula  $\Phi$  in  $S$  such that  $c(y_1, \dots, y_\ell)$  if and only if  $\exists x_1, \dots, x_k : \Phi(y_1, \dots, y_\ell, x_1, \dots, x_k)$ . In that case,  $\Phi$  is called a *primitive positive formula*, or just *pp-formula*. For instance, we can express multiplication using squaring and addition as follows:

$$x \cdot y = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2.$$

This translates into a pp-formula as follows.  $\exists A_0, A_1, A_2, B_0, B_1, B_2 :$

$$\begin{aligned} A_0 &= x + y, & x &= y + B_0, \\ A_0 &= A_1 + A_1, & B_0 &= B_1 + B_1, & A_2 &= B_2 + z. \\ A_2 &= A_1^2, & B_2 &= B_1^2, \end{aligned}$$

Given a pp-formula, we can reduce a CSP with constraint  $c$  to a CSP with a different signature. Here, we replaced the ternary constraint  $x \cdot y = z$  by the binary constraint  $x^2 = y$ .

Furthermore, there are often some range constraints of the form  $x > 0$ ,  $x \in [1/2, 2]$  or even  $x \in [-\delta, \delta]$ , for some  $\delta = O(n^{-c})$ , where  $n$  is the number of variables. These constraints

can be imposed on either all, or a subset of the variables. This makes the above reduction more involved, as we need to pay attention to the ranges in every step. Range constraints are important as we may only be able to encode variables in a certain limited range. Finally, it may be useful to know some structural properties of the variable constraint graph, like planarity [38].

Overall, those techniques have their limitations. As the reductions rely on an explicit way to express one non-linear constraint by another non-linear constraint and addition, we have to find those identities. To illustrate this, we encourage the reader to find a way to express multiplication (in some range) using  $x^2 + y^2 = 1$  and linear constraints. (We consider the constraint  $x = 1$  to be linear. See Appendix A for the solution.) This gets more tricky when dealing with inequality constraints. For instance, it is not clear how to express multiplication with  $x \cdot y \geq 1$  and  $x^2 + y^2 \geq 1$ . We offer 10 euro to the first person, who is able to find a pp-formula to do so. Note that our theorems imply that those two inequalities together with linear constraints are enough to establish  $\exists\mathbb{R}$ -completeness, but we do not describe a pp-formula. At last, translating a pp-formula into a reduction that respects the range constraints for every variable becomes very tedious and lengthy. Furthermore, it only establishes  $\exists\mathbb{R}$ -completeness for those specific constraints. See Abrahamsen and Miltzow [4] for some of those reductions.

To overcome this limitation, we develop a new technique that establishes  $\exists\mathbb{R}$ -completeness for virtually any one non-linear equality constraint. We extend our results and show that any one convex and any one concave inequality constraint are also sufficient to establish  $\exists\mathbb{R}$ -completeness. See Section 1.3 for a formal description of our results and Section 1.6 for an overview of our techniques.

### 1.3 Results

We focus on the special case with essentially only one addition constraint and any one non-linear constraint, see Definition 1.7. While this may seem like a strong limitation, note that addition constraints are commonly easy to encode. In most applications, the non-linear constraint is the crucial one. Before we introduce the main definition, we first specify more precisely how we define the non-linear constraints.

**DEFINITION 1.5 (Function constraints).** Let  $U \subseteq \mathbb{R}^2$  and let  $f: U \rightarrow \mathbb{R}$  be any function. Now we define two constraints corresponding to  $f$  as

$$\text{EqualZero}(f) = \{(x, y) \in U \mid f(x, y) = 0\} \cup (\mathbb{R}^2 \setminus U),$$

and

$$\text{LargerZero}(f) = \{(x, y) \in U \mid f(x, y) \geq 0\} \cup (\mathbb{R}^2 \setminus U).$$

For convenience, we often use the shorthand notation  $f(x, y) = 0$  and  $f(x, y) \geq 0$ . Note that this definition means that the constraints  $\text{EqualZero}(f)$  and  $\text{LargerZero}(f)$  are satisfied whenever  $(x, y)$  is outside of the domain  $U$  of the function  $f$ . We defined the constraints in



this way as it turns out that it makes the soundness part of the proof for future reductions considerably easier. One simply does not need to worry about solutions leaving the domain. We will show that our difficult instances are actually domain adherent, as we will define below.

**DEFINITION 1.6 (domain adherent).** Let  $\Phi$  be a CCSP formula that contains some function constraints, i.e.  $\text{LargerZero}(f)$  or  $\text{EqualZero}(f)$ . Here  $f$  is a function on the domain  $U \subset \mathbb{R}^2$ . We say a solution  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  is *domain adherent* if for every function constraint on variables  $x_i, x_j$ , we have that  $(x_i, x_j) \in U$ . We say  $\Phi$  is domain adherent if this is true for all solutions.

**DEFINITION 1.7 (Curved equality problem (CE)).** Let  $U \subseteq \mathbb{R}^2$  and let  $f$  be a function  $f: U \rightarrow \mathbb{R}$ . Then we define the signature  $C(f, \delta)$  as

$$C(f, \delta) = \{x + y = z, \text{EqualZero}(f), x \geq 0, x = \delta\}.$$

In the *CE problem*, the input consists of a  $\delta \in \mathbb{R}$  and a constraint formula  $\Phi$  on  $n$  variables. The formula  $\Phi$  corresponds to the structure  $\langle \mathbb{R}, C(f, \delta) \rangle$ , where we are promised that  $V(\Phi) \subseteq [-\delta, \delta]^n$  and that  $V(\Phi)$  is domain adherent. We are asked whether  $V(\Phi) \neq \emptyset$ .

Note that the two promises  $V(\Phi) \subseteq [-\delta, \delta]^n$  and domain adherent, while formally independent, are proven essentially in the same way, by scaling variables sufficiently close to the origin.

We would like to emphasize that we are not having a constraint of the form  $x \in [-\delta, \delta]$  in the signature. The property  $V(\Phi) \subseteq [-\delta, \delta]^n$  will be imposed only by using constraints of the structure  $\langle \mathbb{R}, C(f, \delta) \rangle$ . Thus, we are dealing with so-called promise problems from computational complexity. Note that this promise is difficult to check. We will discuss promise problems again in Section 1.4.

This promise is also the reason why we defined the constraint  $\text{EqualZero}(f)$  to be true everywhere outside of the domain of  $f$ . We anticipate that most applications of our result will likely focus on the case where  $[-\delta, \delta]^2$  is a subset of  $U$ . We will discuss the  $\text{EqualZero}(f)$  constraint in more detail in Section 1.4.

Note that although the problem is called curved equality problem, we make no assumptions on  $f$  as part of the definition. We do this explicitly, as there are various technical ways to formulate those assumptions. Abrahamsen, Adamaszek, and Miltzow [2, 4] essentially showed that CE is  $\exists\mathbb{R}$ -complete for  $f = (x - 1)(y - 1) - 1$ . Here, we generalize this to a wider set of functions  $f$  defined below. Recall that a set  $T$  is a *neighborhood* of a point  $p$  if there is an open set  $S$  with  $p \in S \subseteq T$ .

**DEFINITION 1.8 (Well-behaved, triple algebraic).** A function  $f: U \rightarrow \mathbb{R}$  is *well-behaved* if the following conditions are met.

- $f$  is a  $C^3$ -function, with  $U \subseteq \mathbb{R}^2$  being a neighborhood of  $(0, 0)$ ,

- $f(0, 0) = 0$ , and all partial derivatives  $f_x, f_y, f_{xx}, f_{xy}$  and  $f_{yy}$  are rational in  $(0, 0)$ .
- $f_x(0, 0) \neq 0$  or  $f_y(0, 0) \neq 0$ .

A function  $f: U \rightarrow \mathbb{R}$  is *triple algebraic* if each of the three sets  $U, \{(x, y) \in U \mid f(x, y) = 0\}$  and  $\{x, y \in U \mid f(x, y) \geq 0\}$  is semi-algebraic.

Note that if  $p(x, y)$  is a polynomial of the form  $\sum_{i,j} a_{i,j} x^i y^j$ , then  $p$  is well-behaved if and only if  $a_{0,0} = 0, a_{1,0}, a_{0,1}, a_{2,0}, a_{1,1}, a_{0,2}$  are rational, and at least one of  $a_{1,0}$  and  $a_{0,1}$  is nonzero. We want to point out that some readers might find it easier to think of  $U$  as a disk with a small radius. To see that this is equally strong note that a disk around the origin is also a neighborhood of the origin and also a semi-algebraic set. (We need to ask for the radius to be an algebraic number.) Thus the  $\exists\mathbb{R}$ -completeness also works for the case that  $U$  is such a disk. But also the  $\exists\mathbb{R}$ -completeness for disks implies the  $\exists\mathbb{R}$ -completeness for neighborhoods. Although a formal proof is a bit tedious the intuition is that we can restrict the range of the variables to lie within the disk given by the neighborhood condition. We decided to use the language of neighborhoods, instead of disks, as we find it more convenient to work with neighborhoods, at the cost of being a bit more abstract than absolutely necessary.

**DEFINITION 1.9 (Curved).** Let  $f: U \rightarrow \mathbb{R}$  be a function that is well-behaved. We write the curvature of  $f$  at zero by

$$\kappa = \kappa(f) = \left( \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2 + f_y^2)^{\frac{3}{2}}} \right) (0, 0),$$

see Figure 1 for an illustration. We say  $f$  is

- **curved** if  $\kappa(f) \neq 0$ ,
- **convexly curved** if  $\kappa(f) < 0$ , and
- **concavely curved** if  $\kappa(f) > 0$ .

Note that the magnitude of  $\kappa$  equals the inverse of the radius of the osculating circle of the curve  $\{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}$  at the origin, see Figure 3. This is the circle which approximates the curve as close as possible. The sign of  $\kappa$  indicates on which side the osculating circle touches the curve. It is positive if this is on the side where  $f$  is negative, negative if the circle touches on the side where  $f$  is positive, and zero if the origin is an inflection point and the osculating circle is a line.

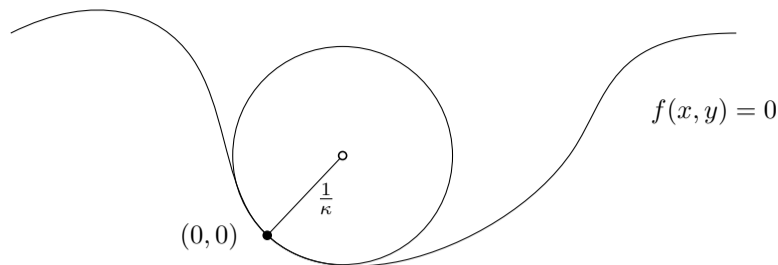
Note that we can define the simpler expression  $\kappa' = \kappa'(f)$

$$\kappa'(f) = \left( f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} \right) (0, 0),$$

and it holds that  $\text{sign}(\kappa) = \text{sign}(\kappa')$ . For this reason, we will work with  $\kappa'$  instead of  $\kappa$ .

Consider a polynomial  $p$  of the form  $p(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ . Then  $\kappa'(p)$  equals

$$\kappa'(p) = a_{01}^2 2a_{20} - 2a_{10} a_{01} a_{11} + a_{10}^2 2a_{02}.$$



**Figure 3.** The formula for  $\kappa$  describes the inverse of the radius of the osculating circle touching  $(0, 0)$  on the curve defined by  $f$ .

In order to define the possible domain of  $\delta$ , we still need one more definition.

**DEFINITION 1.10.** We say a function  $T: \mathbb{N} \rightarrow \mathbb{Q}_{>0}$  is *bounded* if there is a constant  $C$  such that  $T(n) \leq C$ , for all  $n$ . The function  $T$  is referred to as *niceily computable* if  $T(n)$  can be expressed as a fraction of integers represented in binary, and this representation is computable in time polynomial to the size of the integer representation of  $n$ .

We will create instances with  $\delta = T(n)$  as input. Some functions that satisfy these conditions are  $T(n) = 1$ ,  $T(n) = n^{-c}$ , for some fixed constant  $c$ , or the function  $T(n) = 2^{-n}$ .

Now, we are ready to state our main theorem for equality constraints.

**THEOREM 1.11.** *Let  $f: U \rightarrow \mathbb{R}$  be a function that is well-behaved, curved, and triple algebraic. Let  $T$  be a function that is both bounded and niceily computable. In this setting, CE is  $\exists\mathbb{R}$ -complete, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

Note that  $\exists\mathbb{R}$ -membership follows from the fact that  $U$  and  $\{x, y \in \mathbb{R}^2 \mid f(x, y) = 0\}$  are semi-algebraic. Therefore, there must be ETR formulas  $\varphi_U$  and  $\varphi_f$  such that the following two statements hold.

- $\varphi_U(x, y)$  is true if and only if  $(x, y) \in U$ .
- $\varphi_f(x, y)$  is true if and only if  $(x, y) \in \{x, y \in U \mid f(x, y) = 0\}$ .

Let  $\Phi$  be a CE-formula. We replace each occurrence of  $\text{EqualZero}(f)(x, y)$  in  $\Phi$  by  $\varphi_f(x, y) \vee \neg\varphi_U(x, y)$ . This gives us a new equivalent ETR formula  $\Phi'$ . And thus CE is in  $\exists\mathbb{R}$ . Note that the fact that  $f$  is triple algebraic is not needed for the  $\exists\mathbb{R}$ -hardness part of Theorem 1.11.

The motivation for this article was to give a convenient tool to show  $\exists\mathbb{R}$ -hardness of geometric packing. Unfortunately, we are only capable of encoding inequality constraints in geometric packing. Thus, in order to apply our techniques to geometric packing, we adapt Theorem 1.11 to inequality constraints. In the following we define the *convex concave inequality problem* (CCI), which is completely analogous to CE with one subtle difference. The constraint  $f(x, y) = 0$  is replaced by  $f(x, y) \geq 0$  and  $g(x, y) \geq 0$ . The curved constraint  $f(x, y) = 0$  is replaced by convexly curved and concavely curved conditions  $f(x, y) \geq 0$  and  $g(x, y) \geq 0$ .

**DEFINITION 1.12 (Convex concave inequality problem (CCI)).** Let  $U \subseteq \mathbb{R}^2$  and let  $f, g$  be functions such that  $f, g: U \rightarrow \mathbb{R}$ . Then we define the signature  $C(f, g, \delta)$  as

$$C(f, g, \delta) = \{x + y = z, \text{LargerZero}(f), \text{LargerZero}(g), x \geq 0, x = \delta\}.$$

In the *CCI problem*, the input consists of a  $\delta \in \mathbb{R}$  and a constraint formula  $\Phi$  on  $n$  variables. The formula  $\Phi$  corresponds to the structure  $\langle \mathbb{R}, C(f, g, \delta) \rangle$ , where we are promised that  $V(\Phi) \subseteq [-\delta, \delta]^n$  and domain adherent. We are asked whether  $V(\Phi) \neq \emptyset$ .

**THEOREM 1.13.** *Let  $f, g: U \rightarrow \mathbb{R}$  be well-behaved and triple algebraic. Furthermore, let  $f, g$  be respectively convexly curved and concavely curved. Let  $T$  be bounded and nicely computable. In this setting, CCI is  $\exists\mathbb{R}$ -complete, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

To show that CCI is in  $\exists\mathbb{R}$  goes along the same lines as the proof that CE is in  $\exists\mathbb{R}$ . Again, the fact that  $f$  is triple algebraic is not needed for the  $\exists\mathbb{R}$ -hardness part of Theorem 1.13.

## 1.4 Discussion

Theorem 1.11 and Theorem 1.13 are strong generalizations of the  $\exists\mathbb{R}$ -completeness of ETR-INV. The problem ETR-INV was instrumental in establishing  $\exists\mathbb{R}$ -completeness for both the Art Gallery problem [2] and the conference version of the proof for geometric packing [5].

One of the major obstacles of the  $\exists\mathbb{R}$ -completeness proofs of the Art Gallery problem was to find a way to encode inversion. If the authors had known Theorem 1.11 back then, it would have been sufficient to encode essentially any well-behaved and curved constraint on two variables, which is much easier. In this section, we discuss strengths, limitations and different perspectives with respect to our main results.

**Comparison of Main Theorems.** In order to compare Theorem 1.11 and Theorem 1.13, consider the following two signatures and their interpretation for some given well-behaved and curved  $f$ :

$$C_1 = \{x + y = z, x \geq 0, x = \delta, \text{EqualZero}(f)\},$$

and

$$C_2 = \{x + y = z, x \geq 0, x = \delta, \text{LargerZero}(f), \text{LargerZero}(-f)\}.$$

Clearly,  $C_2$  is more expressive than  $C_1$ . Therefore,  $\exists\mathbb{R}$ -hardness of CE implies  $\exists\mathbb{R}$ -hardness of CCI in the special case  $g = -f$ . For unrelated  $f$  and  $g$  there is no further relation between the two theorems. However, in the special case of  $f = y - \bar{f}(x)$  and  $g = \bar{f}(x) - y$ ,  $\exists\mathbb{R}$ -hardness of CCI implies  $\exists\mathbb{R}$ -hardness of CE as follows: we can encode each constraint of the form  $f(x, y) = y - \bar{f}(x) \geq 0$  using the new constraints  $f(x, z_1) = z_1 - \bar{f}(x) = 0, z_2 = y - z_1$ , and  $z_2 \geq 0$ . Similarly, constraints of the form  $g(x, y) = \bar{f}(x) - y \geq 0$  can be encoded in  $C_1$ .

**Promise Problems.** A *promise problem* is defined as an algorithmic problem where the instances are restricted to those which satisfy a certain condition. In other words, we *guarantee* that the condition holds.

In Theorem 1.11, we gave a promise on the problem instance. Namely, we guarantee that the solution set will be contained in a box of a certain size. It is not very common that promises are formulated in this way. However, promise problems are very common and in particular, they can also be treated as decision problems. A prime example is the independent set problem on planar graphs (MISPLANAR). MISPLANAR is known to be NP-complete and it is also a promise problem. The main difference is that we can check whether a graph is planar in polynomial time. However, even if planarity was undecidable the NP-completeness of MISPLANAR would still be valid.

In our scenario, generally, it is not straightforward to verify the promise that the value of each variable lies within the range  $[-\delta, \delta]$ . This makes our result a bit unusual. However, it is relatively straightforward to enforce that all solutions are in the desired range. This follows in two steps. In the first step, we employ a known lemma from the real algebraic geometry literature, which ensures that some solutions must be inside a large ball. Thereafter, we replace each variable by a scaled copy of itself. This will require some small adaption of the constraints. We can then enforce  $-\delta \leq x \leq \delta$ , by the constraints  $(s = x + \delta \text{ and } s \geq 0)$  as well as  $(t + x = \delta \text{ and } t \geq 0)$  without changing the truth value of the instance.

**First-order Theory of the Reals.** With the full first-order theory of the reals it is easier than with CCSPs to define all semi-algebraic sets. Specifically, we can define non-convex sets using only convex constraints, as follows. If we allow a single convex constraint  $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ , then the following formula  $\varphi$  describes the upper half of the boundary of the disk, given by  $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \wedge y \geq 0\}$ :

$$\varphi(x, y) = D(x, y) \wedge \forall_{z \in \mathbb{R}} (D(x, z) \Rightarrow z \leq y).$$

Another way to construct a non-convex constraint using convex constraints is as follows.

$$\varphi(x, y) = D(x, y) \wedge \neg D(x - 1, y)$$

Note that the second example only uses negations and no quantifiers.

Using just the language of CCSPs, it is however impossible to encode such a set using only linear constraints and the constraint  $D(x, y)$ , as any CCSP instance of this form describes a convex set. Note in particular that we may not apply quantifier elimination to the given formula  $\varphi$ , since this is impossible without introducing non-linear constraints different from  $D(x, y)$ . For a more extensive analysis of the semi-algebraic sets which can be described using the first order theory of the reals when the set of atomic formulas is restricted, we refer to [39, 46, 47].

**Convex Constraints.** We want to point out that addition and convexly curved constraints alone seem not to be sufficient to establish  $\exists\mathbb{R}$ -completeness, as convex programs have efficient approximation algorithms [17]. As convex programming is so efficiently fast solvable in practice it would be a big surprise if it would be  $\exists\mathbb{R}$ -complete. However, there are reasons to believe that convex programming is not polynomial-time solvable, see the discussion by O’Donnell [44]. See [45, 6, 59] for an in-depth discussion why convex programming is potentially not polynomial-time solvable.

**Concave Constraints.** When we remove the convex constraint but keep the concave constraint in Theorem 1.13 then we do not know if the problem is  $\exists\mathbb{R}$ -complete. It is easy though to establish NP-hardness in this case [14]. We consider the option that there is another complexity class Concave that characterizes such CCSPs. As with geometric packing with convex pieces, polygonal containers and translations grant the possibility to encode only linear and concave constraints. This problem is a natural candidate to be Concave-complete. We are curious if this intuition could be supported in some mathematically rigorous way.

**Unary Constraints.** Note that the constraint  $x = \delta$  is necessary to ensure that the origin is not always a valid solution. Although  $x \geq 0$  may not be necessary to imply  $\exists\mathbb{R}$ -completeness, our proof heavily relies on it. As an example where this constraint is not needed, consider the case where we have the constraint  $y = x^2$ . In this case we could replace any constraint of the form  $x \geq 0$  by  $x = z^2$ , for some new variable  $z$ . In applications, it is usually very easy to encode unary constraints.

**Binary Constraints.** If we remove the addition constraint, we are left only with constraints in at most two variables. This seems too weak to establish  $\exists\mathbb{R}$ -completeness, as setting  $x$  determines  $y$ , up to finitely many options once we impose the constraint  $f(x, y) = 0$ . On the other hand, very large and irrational solutions can be enforced, which makes it unlikely for those CCSPs to be contained in general in NP. We wonder about the algorithmic complexity of CCSPs with only binary constraints.

**Ternary Constraints.** Given the discussion above, it seems plausible that at least one ternary constraint is required to establish  $\exists\mathbb{R}$ -completeness. Therefore, we find it interesting to focus on ternary constraints. Let’s consider first the natural ternary multiplication constraint  $x \cdot y = z$ . First, we notice that setting all variables to zero satisfies this constraint. If this is the only constraint then we have a polynomial time algorithm. Second, the ternary multiplication constraint  $x \cdot y = z$  can be transformed to the linear constraint  $\log x + \log y = \log z$  [16], in case all variables are positive. This trick can help in case the all-zero solution is not allowed or other unary constraints are introduced. Therefore, the multiplication constraint does not

lead to  $\exists\mathbb{R}$ -hardness by itself. Furthermore, due to the logarithm trick, multiplication seems somewhat easier than other ternary constraints.

It is plausible that this trick or similar tricks can only be applied to exceptional ternary constraints. We leave it as an exciting open problem to determine which ternary constraints lead to  $\exists\mathbb{R}$ -complete CCSPs.

**Arbitrary Constraints.** We want to point out that our results only concern constraints coming from well-behaved functions, instead of allowing arbitrary constraints. Such a restriction is necessary, since otherwise we could, for example, consider CSPs with a constraint that forces a variable to be an integer. This would allow us to encode arbitrary Diophantine equations, making the problem undecidable. Even more Bodirsky and Grohe [13] showed that *any* algorithmic decision problem has an equally difficult CSP problem. As a consequence, any type of classification of continuous constraints must limit the set of allowed constraints in some way.

**Variable-Constraint Graph.** We have completely neglected the variable-constraint incidence graph in this paper. Previous work showed that this graph can be restricted, by self-reduction and a clever application of the addition constraint [25, 38]. We are curious if it is possible to classify hereditary graph classes for which CCI is  $\exists\mathbb{R}$ -complete.

**Universality Results.** Previous reductions of  $\exists\mathbb{R}$ -completeness usually also imply so-called universality results. Giving a proper introduction to universality results is outside the scope of this paper. Universality results translate topological and algebraic phenomena from one type of CSP to another type. See the lecture notes by Matoušek for some introduction to universality theorems in this context [41]. Our methods do not seem to imply these types of universality results. Specifically, if  $f$  is a complicated function that is not even a polynomial, it seems implausible that  $f$  can be used to construct, say,  $\sqrt{2}$ .

**Algebraic Derivatives.** Given the applications that we are aware of, the most complicated part was to check that  $f, g$  and their derivatives are rational at the origin.

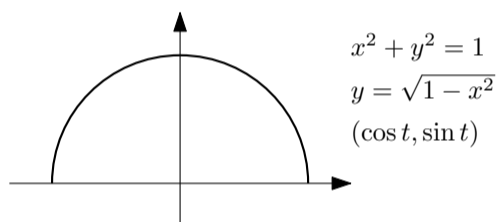
We wonder whether it might be sufficient if the values of  $f, g$  and its derivatives are algebraic at the origin. This weaker condition might follow from some general argument that avoids computing  $f, g$  and its derivatives.

**Constraints true outside of Domain.** We have defined the constraints  $\text{EqualZero}(f)$  and  $\text{LargerZero}(f)$  to be true outside of the domain of  $f$ . We use this formulation to make our results slightly easier to apply. The use case is as follows. Assume that we have some CE instance  $\Phi$  and we are building an instance  $I$  of some other type of, say geometric, problem from it. Assume that we are able to construct a gadget representing a suitable curved and well-behaved function  $f$ . We have to show that  $\Phi$  is a yes instance if and only if  $I$  is a yes instance. One

direction is commonly easy. In case that  $\Phi$  is a yes instance then there must be some  $x$  that satisfies all the constraints. Typically, the construction of  $I$  together with  $x$ , directly shows that  $I$  is also a yes instance. Sometimes, the reverse direction is more difficult. We have to show that if  $I$  is a yes-instance then  $\Phi$  is one as well. The tricky part is that it is sometimes conceivable that  $I$  has a solution, but that this solution could potentially leave the intended range. We still have to show that all constraints of  $\Phi$  are satisfied. This is now trivial for the constraints  $\text{EqualZero}(f)$  and  $\text{LargerZero}(f)$ , as they are defined to be always true outside of the domain of  $f$ . In other words, the way we defined the constraints derived from  $f$  ensures that we do not need to worry about variables leaving their range when applying our results.

## 1.5 Alternative Descriptions

In this subsection, we want to make some comments that might make it easier to apply our results to CCSPs where the constraints are given in explicit form or by a parametrization. Before we delve into technical details consider the following example.



**Figure 4.** Three descriptions of the points on the semi-circle.

**EXAMPLE 1.14.** Let  $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$  be the circle. We can define the function  $f(x, y) = x^2 + y^2 - 1$ , which describes the set  $S$  by  $\{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0\}$ . We can easily check that  $f$  is well-behaved and curved and apply Theorem 1.11. (Note that  $f(0, 0) \neq 0$ , but this can be easily fixed by shifting  $f$ . We do not do this here to keep the notation simple.)

In a different setting, we may know that  $S$  is the graph of a function  $h: [-1, 1] \rightarrow \mathbb{R}$  given by  $h(x) = \sqrt{1 - x^2}$ . This would mean

$$S = \{(x, h(x)) \in \mathbb{R} \times \mathbb{R} \mid x \in [-1, 1]\} \cup \{(x, -h(x)) \in \mathbb{R} \times \mathbb{R} \mid x \in [-1, 1]\}.$$

If we are given such a description, it is possible to rewrite the condition  $y = h(x)$  as  $f(x, y) = 0$  for  $f(x, y) = y - h(x)$ , and we can check whether  $f$  satisfies the necessary conditions. Instead, it turns out we can more easily check the relevant conditions directly on  $h$ .

Another description of the set  $S$  could be by a parametrization  $\gamma(t) = (\cos t, \sin t)$ . With this it holds that  $S = \{\gamma(t) \in \mathbb{R}^2 \mid t \in [-\pi, \pi]\}$ . While we know  $f$  in this specific case, in general, it is not so easy anymore to give an explicit description of  $f$ . We will give some conditions on  $\gamma$  which can be checked to ensure that our theorems can be applied when the constraint has such a parametrized form. ◆



In this section, we derive sufficient conditions to check that  $f$  exists and is well-behaved and curved, even when we do not know how to describe  $f$  explicitly. These conditions are used in at least two applications [5, 36]. In this section, we ignore issues about  $\exists\mathbb{R}$ -membership as they are not so easy to handle and we believe that most readers care about the  $\exists\mathbb{R}$ -hardness part anyways.

**Explicit Description.** We consider the case that the constraint  $f(x, y) = 0$  is described by  $y = h(x)$ .

**DEFINITION 1.15 (well-behaved, curved).** We say  $h: I \rightarrow \mathbb{R}$  is *well-behaved* if it satisfies the following conditions.

- $h$  is a  $C^3$  function, with  $I \subset \mathbb{R}$  being an interval with 0 in its interior.
- $h(0) = 0$ , and  $h'(0)$  and  $h''(0)$  are rational.

We say  $h$  is

- **curved** if  $h''(0) \neq 0$ ,
- **convexly curved** if  $h''(0) > 0$ , and
- **concavely curved** if  $h''(0) < 0$ .

Note that our definition of  $h$  being convexly curved corresponds to  $h$  being a convex function. Note that we have now defined the terms well-behaved, curved etc both for  $f$  as well as for  $h$ . The next lemma justifies this overload, as it shows that they also exactly correspond to one another.

**LEMMA 1.16.** *Let  $h: I \rightarrow \mathbb{R}$  be a well-behaved function and  $f: I \times \mathbb{R} \rightarrow \mathbb{R}$  be defined as  $f(x, y) = y - h(x)$ . Then  $f$  is well-behaved as well. Furthermore, any property (curved, convexly curved, concavely curved) that is satisfied by  $h$  is also satisfied by  $f$ .*

**PROOF.** We check all conditions of well-behavedness one-by-one. We note that  $U = I \times \mathbb{R}$  is indeed a neighborhood of the origin. As  $h$  is  $C^3$ , so is  $f$ . It holds that  $f(0, 0) = 0 - h(0) = 0$ . The derivatives have the following form:

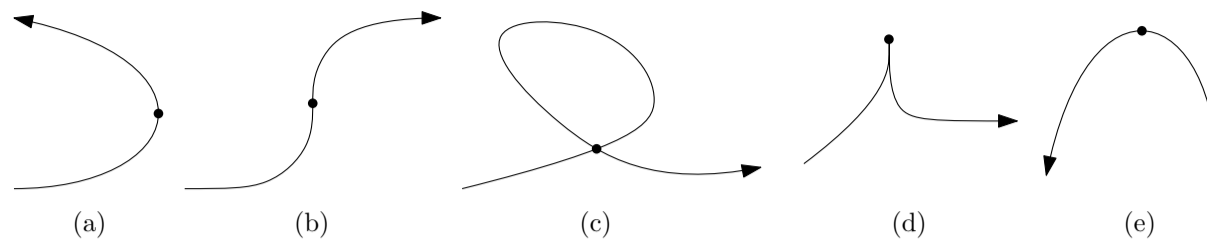
$$f_x = -h', \quad f_y = 1, \quad f_{xx} = -h'', \quad f_{xy} = 0, \quad f_{yy} = 0.$$

Recall that by Young's theorem  $f_{xy} = f_{yx}$ . As  $h(0)$ ,  $h'(0)$ , and  $h''(0)$  are rational, so are the values of  $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$  when evaluated in the origin. We now compute

$$\kappa'(f) = \left( f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} \right) (0, 0) = -h''(0) + 2h'(0) \cdot 0 + (-h'(0))^2 \cdot 0 = -h''(0).$$

Thus the properties are also in correspondence to one another. ■

We are now ready to go to the parametrized description.



**Figure 5.** The curves displayed in (a) to (d) do not satisfy our conditions. In (a) and (b) the situation could be fixed by reversing the role of  $x$  and  $y$ . In (c) the situation could be fixed by restricting the range of  $t$ . Although  $\gamma$  is smooth in (d), the corresponding function  $h = b \circ a^{-1}$  is not. In (e) it is the case that  $a'(t) < 0$ . We can define a new parametrization  $\gamma(-t)$ , which satisfies now all conditions.

**Parameterized Description.** We next consider the case that the set  $S$  that describes the constraint is given by a parametrization  $\gamma = (a, b): I \rightarrow \mathbb{R}^2$ .

**DEFINITION 1.17.** We say a parametrization  $\gamma = (a, b): I \rightarrow \mathbb{R}^2$  is *well-behaved* if it satisfies the following conditions.

- $\gamma$  is a  $C^3$  parametrization, with  $I \subset \mathbb{R}$  being an interval with 0 in its interior.
- $\gamma(t) = (0, 0)$  if and only if  $t = 0$ .
- The functions  $a', b', a'', b''$  are all rational in 0 and  $a'(t) > 0, \forall t \in I$ .

We define  $\kappa'(\gamma) = a'' \cdot b' - b'' \cdot a'(0)$ . We say  $\gamma$  is

- **curved** if  $\kappa'(\gamma) \neq 0$ ,
- **convexly curved** if  $\kappa'(\gamma) < 0$ , and
- **concavely curved** if  $\kappa'(\gamma) > 0$ .

Here, we made the assumptions that  $a'(t) > 0, \forall t \in I$ , so that we can apply the inverse function theorem globally. This makes the notation easier and removes some pathological cases when  $\gamma$  approaches the origin for larger  $t$ . Intuitively, the condition states that  $\gamma$  goes from left to right. Note that if  $a'(t) < 0, \forall t \in I$ , we can just replace the parametrization  $\gamma(t)$  by  $\gamma(-t)$ . It is also not such a strong assumption as  $a'(0) \neq 0$  also implies  $a'(t) \neq 0 \forall t \in J$  for some sufficiently small open interval  $J$  containing zero.

As we will show later, if  $\gamma$  is well-behaved then  $\gamma$  describes the graph of a function  $h(x) = y$ . See Figure 5 for a geometric illustration for the different cases that can occur if one of the conditions is dropped. Specifically, we can describe  $h(x)$  by  $b(a^{-1}(x))$ .

**LEMMA 1.18.** *Let the parametrization  $\gamma$  be well-behaved. Then there is an interval around the origin  $J \subseteq \mathbb{R}$  and some well-behaved  $f$  such that*

$$\{\gamma(t) \in \mathbb{R}^2 \mid t \in I\} = \{(x, y) \in J \times \mathbb{R} \mid f(x, y) = 0\},$$

and such that  $\{(x, y) \in J \times \mathbb{R} \mid f(x, y) \geq 0\}$  is exactly the set of points above the curve given by  $\gamma$ .

Furthermore, it is possible to choose  $f$  such that any property (curved, convexly curved, concavely curved) that is satisfied by  $\gamma$  is also satisfied by  $f$ .

The proof will make use of a specific version of the inverse function theorem that we state here for the benefit of the reader.

**THEOREM 1.19 (Inverse Function Theorem).** *Let  $I \subseteq \mathbb{R}$  be some interval containing 0 in its interior, and let  $a: I \rightarrow \mathbb{R}$  be a  $C^3$ -function, with  $a(0) = 0$  and  $a'(t) \neq 0$  for all  $t \in I$ . In this situation,  $a: I \rightarrow a(I)$  is invertible, and its inverse  $a^{-1}: a(I) \rightarrow I$  is a  $C^3$ -function.*

We want to point out that the function  $a^{-1}$  often cannot be expressed in closed form. For instance, if  $a(t) = t^5 - t - 1$  for  $t \in [1, 2]$ , then  $a'(t)$  is positive for all  $t$ , but  $a^{-1}(0)$  does not admit a closed form expression [61]. And thus it is also not so difficult to find examples of parametrizations for which we cannot find a closed form expression by some function  $f$ . Therefore, it is really useful to have conditions on  $\gamma$  that we can check instead of having to find  $f$ .

**PROOF OF LEMMA 1.18.** First we argue that  $\gamma$  describes the graph of a function  $h$ . Recall that  $\gamma$  consists of the two components  $a$  and  $b$ , i.e.,  $\gamma(t) = (a(t), b(t))$ . We note that all conditions of the inverse function theorem as stated above are satisfied for  $a$ , thus if we let  $J = a(I)$ , then there is inverse function  $a^{-1}: J \rightarrow I$  that is a  $C^3$  function. We now define

$$h(x) = b(a^{-1}(x)),$$

for all  $x \in J$ . Using the fact that  $a^{-1}$  is an inverse of  $a$ , it follows that

$$\{\gamma(t) \in \mathbb{R}^2 \mid t \in I\} = \{(x, h(x)) \in \mathbb{R}^2 \mid x \in J\} = \{(x, y) \in J \times \mathbb{R} \mid y = h(x)\}.$$

If we define  $f(x, y) = y - h(x)$  for  $(x, y) \in J \times \mathbb{R}$ , it follows that

$$\{\gamma(t) \in \mathbb{R}^2 \mid t \in I\} = \{(x, y) \in J \times \mathbb{R} \mid f(x, y) = 0\}.$$

Note that a point  $(x, y)$  in  $J \times \mathbb{R}$  lies above the curve given by  $\gamma$  if and only if  $y \geq h(x)$ , which is equivalent to  $f(x, y) \geq 0$ . This proves the first part of the lemma.

For proving the second part, by Lemma 1.16, it is sufficient to evaluate  $h(x) = b(a^{-1}(x))$  and its derivatives at  $x = 0$ . We start with

$$h'(0) = \frac{b'(a^{-1}(0))}{a'(a^{-1}(0))} = \frac{b'(0)}{a'(0)}.$$

We continue with

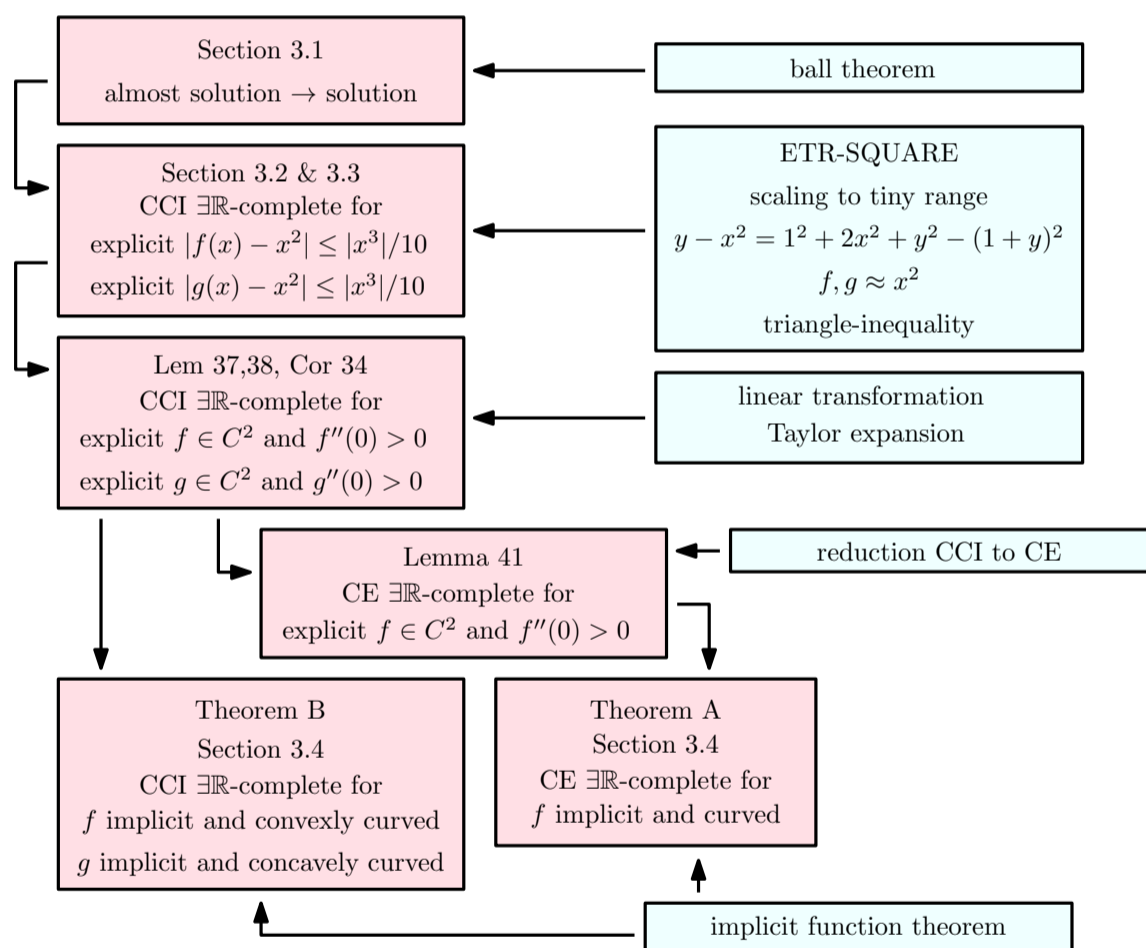
$$h''(0) = \frac{b''(a^{-1}(0))a'(a^{-1}(0)) - a''(a^{-1}(0))b'(a^{-1}(0))}{[a'(a^{-1}(0))]^3} = \frac{b''(0)a'(0) - a''(0)b'(0)}{[a'(0)]^3} = -\frac{\kappa'(\gamma)}{[a'(0)]^3}.$$

Note that, since  $a'(0) > 0$ , this implies that  $h''(0)$  and  $\kappa'(\gamma)$  have opposite signs. This finishes the proof. ■

## 1.6 Proof Overview for CE and CCI

The proofs of Theorem 1.11 and Theorem 1.13 follow several steps which we explain in this section. We start by explaining how  $\exists\mathbb{R}$ -hardness of CE (Theorem 1.11) can be proven, and then we say how this can be modified to prove the hardness of CCI (Theorem 1.13). The structure of the proof is visualized in Figure 6.

**Notation.** In our proofs, newly introduced variables will often be denoted by using double square brackets, like this:  $\llbracket f(x) \rrbracket$ ,  $\llbracket x + y \rrbracket$ ,  $\llbracket x^2 \rrbracket$ , etc. In this notation, formally the whole expression including the brackets and the symbols within it should be understood as the name of the variable, without any special meaning. The symbols within the brackets will usually denote the value which is intuitively represented by the variable.



**Figure 6.** A formal overview of the different steps of the proof to Theorem 1.11 and Theorem 1.13.

**Ball Theorem.** One of the most important tools that we employ is a lemma from real algebraic geometry [8]. It states that for every ETR-formula  $\Phi$  there is a ball  $B$  whose radius only depends on the length  $L$  of  $\Phi$ , such that the following property is satisfied: if  $\Phi$  has at least one solution  $x$  then there must be also a solution  $y$  inside the ball  $B$ . This theorem tells us that solutions cannot get too large. To get an intuition, consider the system of equations  $x_0 = 2, x_{i+1} = x_i^2$ ,

for  $i = 0, 1, \dots, n - 1$ . Clearly,  $x_n = 2^{2^n}$ , which is doubly exponentially large. The ball theorem essentially states that we cannot get much larger numbers.

**Range.** To introduce range constraints is common practice and we inherit them from previous work [4, 5]. We repeat here the argument, for the benefit of the reader. In order to restrict the range of every variable, we first note that the ball theorem already tells us that the range of each variable may be limited by some number  $r$ . We construct  $\varepsilon = \delta/r$  and replace every variable  $x$  by  $\llbracket \varepsilon x \rrbracket = \varepsilon \cdot x$  and consequently we need to adapt all constraints. For instance  $x \cdot y = z$  becomes  $\llbracket \varepsilon x \rrbracket \cdot \llbracket \varepsilon y \rrbracket = \llbracket \varepsilon z \rrbracket \varepsilon$ . In this way, we can easily ensure that if there is a solution at all then there is at least one solution with all variables in the range  $[-\delta, \delta]$ .

We will make use of this re-scaling trick to place all variables in an even smaller range close to zero, as the behavior of  $f$  and  $g$  is better understood close to the origin. Specifically, the error  $|f(x) - x^2| \leq \varepsilon^3$  is small enough to pretend that  $f$  behaves like a squaring function.

**Approximate Solution.** Using the ball theorem, we will establish that equality constraints of the form  $p(x) = 0$  can be slightly weakened to  $|p(x)| \leq \varepsilon$  for some sufficiently small  $\varepsilon$ . To get an intuition consider the following highly simplified cases.

Assume we are given a polynomial equation  $p(x) = 0$ , with  $p \in \mathbb{Z}[X_1, \dots, X_n]$  and we are looking for a solution  $x \in \mathbb{Z}^n$ . Then in particular, we know that for all  $x \in \mathbb{Z}^n$  that  $p(x) \in \mathbb{Z}$ . This readily implies that we can equivalently ask for some  $x \in \mathbb{Z}^n$  that satisfies  $|p(x)| \leq \frac{1}{2}$ . Now, this is trivial for integers as integers have distance at least one to each other. But we can generalize the same principle also to rational and algebraic numbers.

Let  $S = \{\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\}$  be  $n$  rational numbers with  $|a_i|, |b_i| \leq L$ . Thus it is easy to see that  $q, r \in S$  have minimum distance  $\frac{1}{L^2}$ . This implies that if  $|q - r| \leq \frac{1}{L^2}$ , for some  $q, r \in S$ , we can infer that  $q = r$ . Again, this may seem almost trivial, but relies on the simple fact that rational numbers with bounded numerator and denominator have a minimum distance to one another.

Lemma 3.2 generalizes the idea to algebraic numbers. Using the ball theorem, we will establish that algebraic numbers also have some minimum distance to one another, if we restrict their bit-complexity.

**ETR-SQUARE.** We use a theorem by Abrahamsen and Miltzow that shows that ETR-SQUARE is  $\exists\mathbb{R}$ -complete [4]. In this variant, we essentially have only addition ( $x + y = z$ ) and squaring constraints ( $x^2 = y$ ). Furthermore, the range of each variable is restricted to a small range around zero. For the sake of completeness and readability, we present a self-contained proof in Section 2.

**Explicit.** Given those tools, we can show that we can replace a squaring constraint with explicit constraints ( $f(x) = y$ ). We start by only considering  $f$  which satisfy

$$|f(x) - x^2| \leq \frac{1}{10}x^3. \quad (1)$$

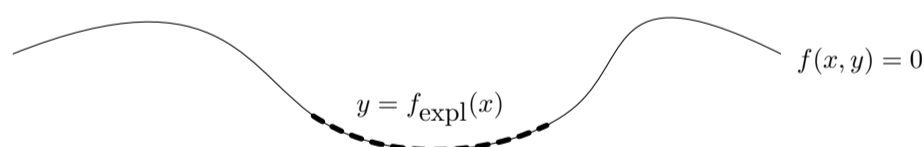
The idea of the reduction from ETR-SQUARE is simple but tedious. We can rewrite the constraint  $x^2 = y$  as a linear combination of squares as follows

$$1^2 + 2x^2 + y^2 - (1 + y)^2 = 0.$$

Now, we can replace each square using the function  $f$  to  $f(1) + 2f(x) + f(y) - f(1 + y) = 0$ . As  $f$  is approximately squaring, this implies that we are approximately enforcing the constraint  $x^2 = y$ . In other words, we enforce  $|x^2 - y| \leq \varepsilon$ . Note that this is the technically most tedious step to make rigorous as we will later see. As we have discussed above it is sufficient to enforce each constraint approximately. The technical difficulty is many-fold. We need to work with scaled variables, instead of the original variables. Furthermore, we have to take into consideration that when we construct  $\varepsilon$  that this also makes the formula longer. In particular, this means that the definition of  $\varepsilon$  cannot depend on the newly constructed instance, but has to depend on the original instance.

Using linear transformations and Taylor expansion on  $f$ , we can replace Condition (1) relatively easily by Condition (2):

$$f \text{ is three times differentiable and } f''(0) > 0. \quad (2)$$



**Figure 7.** The implicit function theorem tells us that there is an function  $f_{\text{expl}}$  such that the curve  $y = f_{\text{expl}}(x)$  is locally identical to the curve  $f(x, y) = 0$ .

**Implicit.** We are now ready to handle the more general case of constraints in implicit form ( $f(x, y) = 0$ ). The implicit function theorem tells us that there is a function  $f_{\text{expl}}$  such that the curve  $y = f_{\text{expl}}(x)$  is locally identical to the curve  $f(x, y) = 0$ , see Figure 7. The properties of the partial derivatives of  $f$  translate to properties of the partial derivatives of  $f_{\text{expl}}$ . In this way, we can infer hardness of the CSP with constraint  $f(x, y) = 0$  from the problem with constraint  $y = f_{\text{expl}}(x)$ .

**Inequalities.** Until this point, we discussed the hardness proof of CE. In CCI, we instead have inequality constraints. The case of inequalities goes analogously to the equality case. We need one convexly curved and one concavely curved inequality. Whenever we want to upper bound an expression, we use one inequality and whenever we need to lower bound something, we use the other one. While on the surface this is not so difficult, it makes the reduction from ETR-SQUARE to CCI considerably more tedious. Specifically, it makes it harder to have an intuition on several technical steps and the meaning of several intermediate variables.

## 2. ETR-Square

This section is dedicated to showing that ETR can be reduced to ETR-SQUARE. We execute all steps of the reduction in great detail for the sake of completeness. This section is largely copied from the paper by Abrahamsen and Miltzow [4]. We mainly simplified the proofs, as we only show  $\exists\mathbb{R}$ -completeness and we leave out the parts that were needed to preserve topological or algebraic properties. Note that large parts in this section can be considered folklore. Similar reductions have been described by Schaefer and Štefankovič [52]. We say two ETR formulas are *equisatisfiable* if they have the same truth value. We say that a formula  $\Phi$  is true if  $V(\Phi)$  is non-empty.

### 2.1 Reduction to Conjunctive Form

**DEFINITION 2.1.** An ETR-CONJ formula  $\Phi = \Phi(x_1, \dots, x_n)$  is a conjunction  $C_1 \wedge \dots \wedge C_m$ , where  $m \geq 0$  and each  $C_i$  is of one of the two forms

$$x \geq 0, \quad p(y_1, \dots, y_l) = 0$$

for  $x, y_1, \dots, y_l \in \{x_1, \dots, x_n\}$  and  $p$  a polynomial.

In the problem ETR-CONJ, we are given an ETR-CONJ formula  $\Phi$ . The goal is to decide if  $V(\Phi)$  is non-empty.

We want to point out that ETR-CONJ is a CSP, but it has an infinite set of possible constraints, one for each polynomial. Thus, we find it inconvenient to use the language of CSPs here. Furthermore, we allow polynomials represented by any well-defined term built from the symbols  $\{0, 1, x_1, \dots, x_n, +, -, \cdot\}$  and brackets, (e.g.  $p = (x + 1)(x - 1) + x$ ), and not just those in standard form (e.g.  $p = x^2 + x - 1$ ). Note that since there are no strict inequalities in a formula  $\Phi$  in ETR-CONJ, the set  $V(\Phi)$  is closed. We show how to reduce a general ETR formula to an ETR-CONJ formula.

**LEMMA 2.2.** *Given an ETR formula  $\Phi$ , we can compute in linear time an equisatisfiable ETR-CONJ formula  $\Psi$ .*

**PROOF.** We start with an ETR formula  $\Phi$  and modify it repeatedly to attain an ETR-CONJ formula  $\Psi$ . Each modification leads to an equisatisfiable formula. Our modifications can be summarized in four steps. (1) Delete “ $\neg$ ”. (2) Delete “ $>$ ”. (3) Delete “ $\geq$ ”. (4) Delete “ $\vee$ ”. In the rest of this proof,  $p$  and  $q$  denote polynomials.

Step (1): Here, we merely “pull” every negation  $\neg$  in front of every atomic predicate. For instance  $\neg(A \vee B \vee C)$  becomes  $(\neg A \wedge \neg B \wedge \neg C)$ . To see that this can be done in linear time, note that the length of  $\Phi$  is at least the number of atomic predicates. At the end of this process, every atomic predicate is preceded by either a negation or not. It may be that  $\wedge$  and  $\vee$  symbols are swapped, but each is counted as one symbol.

Thereafter each atomic predicate preceded by  $\neg$  is replaced as follows:

$$\begin{aligned} \neg(q > 0) &\mapsto -q \geq 0 \\ \neg(q = 0) &\mapsto (q > 0) \vee (-q > 0) \\ \neg(q \geq 0) &\mapsto -q > 0 \end{aligned}$$

Those replacements are done repeatedly until there are no occurrences of “ $\neg$ ” left in the formula.

Step (2): We replace each strict inequality as follows:

$$q > 0 \mapsto (q \cdot y \cdot y - 1 = 0),$$

where  $y$  is a new variable. Those replacements are done repeatedly till there are no occurrences of “ $>$ ” left in the formula.

Step (3): We replace all atomic predicates of the form  $q \geq 0$  by the predicate  $q - z^2 = 0$ , where  $z$  denotes a new variable.

Step (4): We delete disjunctions as follows. It will also be necessary to replace some conjunctions. Let  $\Phi$  be the formula after Step (1)–(3). Let  $\Psi$  be an, initially empty, ETR-CONJ formula. In this step, we will describe an algorithm to repeatedly modify  $\Phi$  and  $\Psi$  in such a way that  $\Phi \wedge \Psi$  stays equisatisfiable to the initial value of  $\Phi$ . We will continue these modifications until  $\Phi$  consists of just a single equation of the form  $p = 0$ .

While  $\Phi$  is not of this form, it either contains a disjunction of the form  $p = 0 \vee q = 0$ , or a conjunction of the form  $p = 0 \wedge q = 0$ . The disjunction  $p = 0 \vee q = 0$  we can replace by a single equation  $p \cdot q = 0$ . For a conjunction  $p = 0 \wedge q = 0$ , we add new variables  $x, y$  and replace it in  $\Phi$  by  $x \cdot x + y \cdot y = 0$ , while we also replace  $\Psi$  by  $\Psi \wedge (p - x = 0) \wedge (q - y = 0)$ . Note that in each step, the number of atomic formulas in  $\Phi$  is reduced by 1, so we know that the reduction terminates in a linear number of steps.

When  $\Phi$  consists of just a single polynomial equation, we can replace  $\Psi$  by  $\Psi \wedge \Phi$ , and we conclude that  $\Psi$  is an ETR-CONJ formula which is equisatisfiable to the original  $\Phi$ .

At first, it might seem easier to replace  $p = 0 \wedge q = 0$  by  $p \cdot p + q \cdot q = 0$ . However, we want our reduction to be linear and the simplified step could, if done repeatedly, lead to very long formulas. With the replacement rules we have suggested, the length of the formula increases



by at most a constant factor. This reduction takes linear time and the final formula  $\Psi$  is an ETR-CONJ formula. ■

## 2.2 Reduction to Compact Semi-Algebraic Sets

In this section, we show the hardness of ETR-COMPACT. In that variant, we are promised that the solution space is compact. To do so, we employ a theorem that states that every solution space is either empty or intersects a large ball.

**DEFINITION 2.3.** In the problem ETR-COMPACT, we are given an ETR-CONJ formula  $\Phi$  with the promise that  $V(\Phi)$  is compact. The goal is to decide if  $V(\Phi)$  is non-empty.

We need a tool from real algebraic geometry. The following corollary has been pointed out by Schaefer and Štefankovič [52] in a simplified form. We always use logarithms with base two.

**COROLLARY 2.4 (Basu, Roy [8] Theorem 2).** *Let  $\Phi$  be an instance of ETR of length  $L \geq 4$  such that  $V(\Phi)$  is a non-empty subset of  $\mathbb{R}^n$ . Let  $B$  be the set of points in  $\mathbb{R}^n$  at distance at most  $2^{L^{8n}} = 2^{2^{8n \log L}}$  from the origin. Then  $B \cap V(\Phi) \neq \emptyset$ .*

**LEMMA 2.5.** *There is a reduction from ETR-CONJ to ETR-COMPACT in  $O(L \log L)$  time, where  $L$  is the length of the formula.*

**PROOF.** Let an instance  $\Phi$  of ETR-CONJ be given and define  $k = \lceil 8n \log L \rceil$ . To make an equisatisfiable formula  $\Psi$  such that  $V(\Psi)$  is compact, we start by including all the variables and constraints of  $\Phi$  in  $\Psi$ . We then construct the variables  $\llbracket 2^{2^0} \rrbracket, \dots, \llbracket 2^{2^k} \rrbracket$ , which will always take the values  $2^{2^0}, \dots, 2^{2^k}$  respectively. We use repeated squaring as follows.

$$\begin{aligned} \llbracket 2^{2^0} \rrbracket - 1 - 1 &= 0 \\ \llbracket 2^{2^1} \rrbracket - \llbracket 2^{2^0} \rrbracket \cdot \llbracket 2^{2^0} \rrbracket &= 0 \\ &\vdots \\ \llbracket 2^{2^k} \rrbracket - \llbracket 2^{2^{k-1}} \rrbracket \cdot \llbracket 2^{2^{k-1}} \rrbracket &= 0 \end{aligned}$$

For each variable  $x$  of  $\Phi$ , we now introduce the variables  $\llbracket x + 2^{2^k} \rrbracket$  and  $\llbracket 2^{2^k} - x \rrbracket$  and the constraints

$$\begin{aligned} \llbracket x + 2^{2^k} \rrbracket - x - \llbracket 2^{2^k} \rrbracket &= 0 \\ \llbracket x + 2^{2^k} \rrbracket &\geq 0 \\ \llbracket 2^{2^k} - x \rrbracket - \llbracket 2^{2^k} \rrbracket + x &= 0 \\ \llbracket 2^{2^k} - x \rrbracket &\geq 0. \end{aligned}$$

Note that this corresponds to introducing the constraint  $-2^{2^k} \leq x \leq 2^{2^k}$  in  $\Psi$ .

Observe that the ball  $B$  centered around the origin with radius  $2^{2^k}$  is contained in the cube  $\left[-2^{2^k}, 2^{2^k}\right]^n$ , so  $B \cap V(\Phi) \subseteq V(\Psi)$ . It now follows by Corollary 2.4 that

$$V(\Phi) \neq \emptyset \Leftrightarrow V(\Psi) \neq \emptyset.$$

Note that  $V(\Psi)$  is compact since  $\Psi$  contains no strict inequalities and each variable is bounded. This finishes the proof. ■

### 2.3 Reduction to ETR-AMI

In this section we will show  $\exists\mathbb{R}$ -hardness of the problem ETR-AMI. The term ETR-AMI is an abbreviation for *Existential Theory of the Reals with Addition, Multiplication, and Inequalities*.

**DEFINITION 2.6 (ETR-AMI).** We define the set of constraints  $C_{\text{AMI}}$  as

$$C_{\text{AMI}} = \{x + y = z, x \cdot y = z, x \geq 0, x = 1\}$$

Now we define the ETR-AMI problem as the CCSP given by  $C_{\text{AMI}}$ .

**LEMMA 2.7 (ETR-AMI Reduction).** *Given an instance of ETR-COMPACT defined by a formula  $\Phi$ , we can in  $O(|\Phi|)$  time construct an equisatisfiable ETR-AMI formula  $\Psi$  such that  $V(\Psi)$  is compact.*

**PROOF.** Recall that  $\Phi$  is a conjunction of atomic formulas of the form  $p = 0$  for a polynomial  $p$  and  $x \geq 0$  for a variable  $x$ . Each polynomial  $p$  may contain minuses, zeros, and ones. The reduction has four steps. In each step, we make changes to  $\Phi$ . In the end,  $\Phi$  has become a formula  $\Psi$  with the desired properties. In step (1)–(3), we remove unwanted ones, zeros, and minuses by replacing them with constants. In step (4), we eliminate complicated polynomials.

Step (1): We introduce the constant variable  $\llbracket 1 \rrbracket$  and the constraint  $\llbracket 1 \rrbracket = 1$  to  $\Phi$ . We then replace all appearances of 1 with  $\llbracket 1 \rrbracket$  in the atomic formulas of the form  $p = 0$ .

Step (2): We introduce the constant variable  $\llbracket 0 \rrbracket$  and the constraint  $\llbracket 1 \rrbracket + \llbracket 0 \rrbracket = \llbracket 1 \rrbracket$  to  $\Phi$ . We then replace all appearances of 0 with  $\llbracket 0 \rrbracket$  except in the constraints of the form  $x \geq 0$ .

Step (3): We introduce the constant variable  $\llbracket -1 \rrbracket$  and the constraint  $\llbracket 1 \rrbracket + \llbracket -1 \rrbracket = \llbracket 0 \rrbracket$  to  $\Phi$ . We then replace all appearances of minus with a multiplication by  $\llbracket -1 \rrbracket$  in  $\Phi$ .

Step (4): We replace bottom-up every occurrence of multiplication and addition by a new variable and an extra addition or multiplication constraint. Here are two examples of such replacements:

$$\begin{aligned} (x_1 + x_2 \cdot x_4 + x_5) \cdot x_6 = \llbracket 0 \rrbracket &\mapsto (x_1 + z_1 + x_5) \cdot x_6 = \llbracket 0 \rrbracket \wedge z_1 = x_2 \cdot x_4 \\ (x_1 + z_1 + x_5) \cdot x_6 = \llbracket 0 \rrbracket &\mapsto (z_2 + x_5) \cdot x_6 = \llbracket 0 \rrbracket \wedge z_2 = x_1 + z_1. \end{aligned}$$

In this way, every atomic predicate is eventually transformed to atomic predicates of ETR-AMI or is of the form  $x = \llbracket 0 \rrbracket$ . In the latter case, we replace  $x = \llbracket 0 \rrbracket$  by  $x + \llbracket 0 \rrbracket = \llbracket 0 \rrbracket$ .

To see that the reduction is linear, note that every replacement adds a constant to the length of the formula. Furthermore, at most linearly many replacements will be done. All the above steps preserve the truth value of the formula and the compactness of the solution set. ■

## 2.4 Reduction to ETR-SMALL

In this section, we show the hardness of ETR-SMALL, as defined below. The reduction works in two steps. In the first step, we create a very small number using repeated squaring and in the second step, we scale every variable to be in the correct range.

**DEFINITION 2.8 (ETR-SMALL).** We define the set of constraints  $C_{\text{SMALL}}$  as

$$C_{\text{SMALL}} = \{x + y = z, x \cdot y = z, x \geq 0, x = \frac{1}{2}\}$$

Now we define the ETR-SMALL problem as the CCSP given by  $C_{\text{SMALL}}$ . Furthermore, for every instance  $\Phi$  we are promised that  $V(\Phi) \subseteq [-\frac{1}{2}, \frac{1}{2}]^n$ .

We are going to present a reduction from the problem ETR-AMI to ETR-SMALL. As a preparation, we present another tool from real algebraic geometry. Schaefer [51] made the following simplification of a result from [8], which we will use. More refined statements can be found in [8].

**COROLLARY 2.9 ([8]).** *If a bounded semi-algebraic set in  $\mathbb{R}^n$  has bit-complexity at most  $L \geq 5n$ , then all its points have distance at most  $2^{2^{L+5}}$  from the origin.*

**LEMMA 2.10 (ETR-SMALL Reduction).** *Given an ETR-AMI formula  $\Phi$  such that  $V(\Phi)$  is compact, we can in  $O(|\Phi|)$  time construct an equisatisfiable instance of ETR-SMALL.*

**PROOF.** Let  $\Phi$  be an instance of ETR-AMI with  $n$  variables  $x_1, \dots, x_n$ . We construct an instance  $\Psi$  of ETR-SMALL.

We set  $\varepsilon = 2^{-2^{L+6}}$ , where  $L = |\Phi|$ . In  $\Psi$ , we first define a constant variable  $\llbracket \varepsilon \rrbracket$ . This is obtained by exponentiation by squaring, using  $O(L)$  new constant variables and constraints. We first define  $\llbracket 0 \rrbracket$ , and  $\llbracket 2^{-2^0} \rrbracket$ , i.e.  $1/2$ , by the equations

$$\begin{aligned} \llbracket 2^{-2^0} \rrbracket &= \frac{1}{2} \\ \llbracket 0 \rrbracket + \llbracket 2^{-2^0} \rrbracket &= \llbracket 2^{-2^0} \rrbracket \end{aligned}$$

We then use the following equations for all  $i \in \{0, \dots, L+5\}$ ,

$$\llbracket 2^{-2^i} \rrbracket \cdot \llbracket 2^{-2^i} \rrbracket = \llbracket 2^{-2^{i+1}} \rrbracket$$

Finally, we define  $\llbracket \varepsilon \rrbracket$  by the constraint  $\llbracket \varepsilon \rrbracket + \llbracket 0 \rrbracket = \llbracket 2^{-2^{L+6}} \rrbracket$ .

In  $\Psi$ , we use the variables  $\llbracket \varepsilon x_1 \rrbracket, \dots, \llbracket \varepsilon x_n \rrbracket$  instead of  $x_1, \dots, x_n$ . An equation of  $\Phi$  of the form  $x = 1$  is transformed to the equation  $\llbracket \varepsilon x \rrbracket + \llbracket 0 \rrbracket = \llbracket \varepsilon \rrbracket$  in  $\Psi$ . An equation of  $\Phi$  of the form  $x + y = z$  is transformed to the equation  $\llbracket \varepsilon x \rrbracket + \llbracket \varepsilon y \rrbracket = \llbracket \varepsilon z \rrbracket$  of  $\Psi$ . For an equation of  $\Phi$  of the form  $x \cdot y = z$ , we also introduce a variable  $\llbracket \varepsilon^2 z \rrbracket$  of  $\Psi$  and the equations

$$\begin{aligned} \llbracket \varepsilon x \rrbracket \cdot \llbracket \varepsilon y \rrbracket &= \llbracket \varepsilon^2 z \rrbracket \\ \llbracket \varepsilon \rrbracket \cdot \llbracket \varepsilon z \rrbracket &= \llbracket \varepsilon^2 z \rrbracket. \end{aligned}$$

At last, constraints of the form  $x \geq 0$  become  $\llbracket \varepsilon x \rrbracket \geq 0$ .

We now describe a function  $f : V(\Phi) \rightarrow V(\Psi)$  in order to show that  $\Phi$  and  $\Psi$  are equisatisfiable. Let  $\mathbf{x} = (x_1, \dots, x_n) \in V(\Phi)$ . In order to define  $f$ , it suffices to specify the values of the variables of  $\Psi$  depending on  $\mathbf{x}$ . For all the constant variables  $\llbracket 2^{-2^0} \rrbracket, \llbracket 2^{-2^1} \rrbracket, \llbracket 2^{-2^2} \rrbracket, \dots$ , we define them in the natural way as  $\llbracket 2^{-2^i} \rrbracket = 2^{-2^i}$  and  $\llbracket \varepsilon \rrbracket = 2^{-2^{L+6}}$ . For all  $i \in \{1, \dots, n\}$ , we now define  $\llbracket \varepsilon x_i \rrbracket = \varepsilon x_i$  and (when  $\llbracket \varepsilon^2 x_i \rrbracket$  appears in  $\Psi$ )  $\llbracket \varepsilon^2 x_i \rrbracket = \varepsilon^2 x_i$ . Since  $\mathbf{x}$  is a solution to  $\Phi$ , it follows from the constraints of  $\Psi$  that these assignments are a solution to  $\Psi$ .

We need to verify that  $\Psi$  defines an ETR-SMALL problem, i.e., that  $\Psi$  satisfies the promise that  $V(\Psi) \subset [-\frac{1}{2}, \frac{1}{2}]^m$ , where  $m$  is the number of variables of  $\Psi$ . To this end, consider an assignment of the variables of  $\Psi$  that satisfies all the constraints. Note first that the constant variables are non-negative and at most  $\frac{1}{2}$ . For the other variables, we consider the inverse  $f^{-1}$ , which is given by the assignment  $x_i = \llbracket \varepsilon x_i \rrbracket / \varepsilon$  for all  $i \in \{1, \dots, n\}$ . It follows that this yields a solution to  $\Phi$ . Since  $V(\Phi)$  is compact, it follows from Corollary 2.9 that  $|\llbracket \varepsilon x_i \rrbracket / \varepsilon| \leq 2^{2^{L+5}}$ . Hence  $|\llbracket \varepsilon x_i \rrbracket| \leq \varepsilon \cdot 2^{2^{L+5}} = 2^{-2^{L+6}} \cdot 2^{2^{L+5}} \leq \frac{1}{2}$ . Similarly, when  $\llbracket \varepsilon^2 x_i \rrbracket$  is a variable of  $\Psi$ , we get  $|\llbracket \varepsilon^2 x_i \rrbracket| \leq \varepsilon < \frac{1}{2}$ .

By the existence of  $f$  and  $f^{-1}$ , we have now established that  $V(\Phi) \neq \emptyset$  if and only if  $V(\Psi) \neq \emptyset$ . In other words,  $\Phi$  and  $\Psi$  are equisatisfiable. The length of  $\Psi$  is  $O(L)$  longer than the length of  $\Phi$ , and  $\Psi$  can thus be computed in  $O(|\Phi|)$  time. ■

## 2.5 Reduction to ETR-SQUARE

From here, we can prove the hardness of the following problem:

**DEFINITION 2.11 (ETR-SQUARE).** We define the set of constraints  $C_{\text{SQUARE}}$  as

$$C_{\text{SQUARE}} = \{x + y = z, y = x^2, x \geq 0, x = 1\}.$$

Now we define the ETR-SQUARE problem as the CCSP given by  $C_{\text{SQUARE}}$ . Furthermore, for every instance  $\Phi$  we are promised that  $V(\Phi) \subseteq [-1, 1]^n$ .

**LEMMA 2.12 (ETR-SQUARE Reduction).** *Given an instance  $\Phi$  of ETR-SMALL, we can in  $O(|\Phi|)$  time construct an equisatisfiable instance of ETR-SQUARE.*

**PROOF.** We start with an ETR-SMALL instance  $\Phi$ . To this instance we add a variable  $\llbracket 1 \rrbracket$  and a constraint  $\llbracket 1 \rrbracket = 1$ . Next we replace every constraint of the form  $x = \frac{1}{2}$  by a constraint  $x + x = \llbracket 1 \rrbracket$ . Finally, for every constraint of the form  $x \cdot y = z$ , we introduce the following new variables:

$$\llbracket x^2 \rrbracket, \llbracket y^2 \rrbracket, \llbracket x + y \rrbracket, \llbracket (x + y)^2 \rrbracket, \llbracket x^2 + 2xy \rrbracket, \llbracket 2xy \rrbracket, \llbracket xy \rrbracket$$

and we add the following constraints:

$$\begin{aligned} \llbracket x^2 \rrbracket &= x^2 \\ \llbracket y^2 \rrbracket &= y^2 \\ \llbracket x + y \rrbracket &= x + y \\ \llbracket (x + y)^2 \rrbracket &= \llbracket x + y \rrbracket^2 \\ \llbracket (x + y)^2 \rrbracket &= \llbracket x^2 + 2xy \rrbracket + \llbracket y^2 \rrbracket \\ \llbracket x^2 + 2xy \rrbracket &= \llbracket x^2 \rrbracket + \llbracket 2xy \rrbracket \\ \llbracket 2xy \rrbracket &= \llbracket xy \rrbracket + \llbracket xy \rrbracket \\ \llbracket xy \rrbracket &= z. \end{aligned}$$

Every constraint of the form  $x + y = z$  or  $x \geq 0$  is not changed. After performing all these changes, which only needs linear time, we have an ETR-SQUARE formula  $\Psi$ . Furthermore, it can be checked that every solution of this ETR-SQUARE formula corresponds uniquely to a solution of the original ETR-SMALL formula. Also the fact that  $V(\Phi) \subseteq [-1/2, 1/2]^n$  can be seen to imply that  $V(\Psi) \subseteq [1, 1]^m$ , where  $m$  is the number of variables in  $\Psi$ . This proves that  $\Psi$  is an ETR-SQUARE instance which is equisatisfiable to the original ETR-SMALL instance. Therefore the reduction is valid. ■

### 3. Proof of CCSP-Theorems

In this section, we will prove Theorem 1.11 and Theorem 1.13.

#### 3.1 Approximate Solutions

We start by proving a lemma which plays an important role in the proofs of Theorem 1.11 and Theorem 1.13. The following lemma intuitively states the following: if an ETR-SQUARE formula  $\Phi$  has something which is “almost a solution”, with an error of at most  $2^{-2^{O(|\Phi|)}}$ , then  $\Phi$  also admits an actual solution. Similar results were established in [24].

**DEFINITION 3.1.** Let  $\Phi = \Phi(x_1, \dots, x_n)$  be an ETR-SQUARE formula such that  $V(\Phi) \subseteq [-1, 1]^n$ . For  $\varepsilon \geq 0$ , define  $\Phi_\varepsilon$  as the formula where every constraint of the form  $y = x^2$  is replaced by a constraint of the form  $|y - x^2| \leq \varepsilon$ , and where constraints  $-1 \leq x \leq 1$  are added for every  $x \in \{x_1, \dots, x_n\}$ .

**LEMMA 3.2.** Let  $\Phi = \Phi(x_1, \dots, x_n)$  be an ETR-SQUARE formula such that  $V(\Phi) \subseteq [-1, 1]^n$ . Now there exists an  $M \in \mathbb{Z}$  with  $M = O(|\Phi|)$  and  $\varepsilon = 2^{-2^M}$ , such that  $\Phi$  and  $\Phi_\varepsilon$  are equisatisfiable.

We use the following result from Schaefer and Štefankovič [52], see also [33]:

**COROLLARY 3.3 (Corollary 3.4 from [52]).** If two semi-algebraic sets in  $\mathbb{R}^n$  each of bit-complexity at most  $L \geq 5n$  have positive distance (for example, if they are disjoint and compact), then that distance is at least  $2^{-2^{L+5}}$ .

Here, the distance between two subsets  $X, Y \subseteq \mathbb{R}^n$  is defined as  $\inf_{x \in X, y \in Y} d(x, y)$ . Note that in the case where  $X$  and  $Y$  are compact, the infimum in this definition may be replaced by a minimum.

**PROOF OF LEMMA 3.2.** First note that for any  $\varepsilon \geq 0$ , we have  $V(\Phi) \subseteq V(\Phi_\varepsilon)$ . In particular, if  $V(\Phi)$  is nonempty, then also any  $V(\Phi_\varepsilon)$  is nonempty. For the rest of the proof, we will assume that  $V(\Phi)$  is empty, and construct  $M$  and  $\varepsilon = 2^{-2^M}$  such that  $V(\Phi_\varepsilon)$  is also guaranteed to be empty.

Suppose that  $\Phi$  has  $n$  variables, and contains  $r$  constraints of the form  $y = x^2$ . For every  $\varepsilon \geq 0$ , we define  $\Phi'_\varepsilon$  as the formula on variables  $x_1, \dots, x_n, \eta_1, \dots, \eta_r$  obtained from  $\Phi$  by replacing every constraint of the form  $y = x^2$  by constraints  $y = x^2 + \eta_i$  and  $-\varepsilon \leq \eta_i \leq \varepsilon$ , and where constraints  $-1 \leq x \leq 1$  are added for every  $x \in \{x_1, \dots, x_n\}$ . Note that there is a natural bijection between  $V(\Phi_\varepsilon)$  and  $V(\Phi'_\varepsilon)$  for every  $\varepsilon \geq 0$ . Since we assumed  $V(\Phi) = \emptyset$ , it also follows that  $V(\Phi'_0) = \emptyset$ . We furthermore define  $\Phi'_\infty$  in the same way, except that we drop the constraints of the form  $-\varepsilon \leq \eta_i \leq \varepsilon$ . Observe that  $V(\Phi'_\infty)$  is bounded: the fact that every variable  $x_i$  is bounded by 1 in absolute value, implies that every variable  $\eta_i$  is bounded by 2 in absolute value. In particular  $V(\Phi'_\infty)$  is compact.

Next define  $\Psi$  to be the formula on the same variables  $x_1, \dots, x_n, \eta_1, \dots, \eta_r$  which enforces  $\eta_i = 0$  for all  $1 \leq i \leq r$  and  $-1 \leq x_i \leq 1$  for all  $1 \leq i \leq n$ . Note that  $V(\Phi'_\infty) \cap V(\Psi) = V(\Phi'_0) = \emptyset$ , and furthermore that both  $V(\Phi'_\infty)$  and  $V(\Psi)$  are compact and nonempty. This means that we can apply Corollary 3.3. Let  $L$  be the maximum of  $5n$  and the bit complexities of  $\Phi'_\infty$  and  $\Psi$ . Note that  $L$  is linear in the length of  $\Phi$ . We conclude that the distance between  $V(\Phi'_\infty)$  and  $V(\Psi)$  is at least  $2^{-2^{L+5}}$ , by Corollary 3.3.

Recall that  $r$  is the number of constraints of the form  $y = x^2$  in  $\Phi$ . Setting  $M = L+6$ , it can be shown that  $r \cdot 2^{-2^M} < 2^{-2^{L+5}}$ . Specifically,  $M$  is linear in the length of  $\Phi$ . We take  $\varepsilon = 2^{-2^M}$ . Suppose, for the purpose of contradiction, that  $V(\Phi_\varepsilon) \neq \emptyset$ , and therefore also  $V(\Phi'_\varepsilon) \neq \emptyset$ . Let  $P \in V(\Phi'_\varepsilon)$ ,

and let  $P'$  be the point we obtain by setting all the  $\eta_i$ -coordinates of  $P$  to 0. Now  $P'$  is contained in  $V(\Psi)$ . Since every  $\eta_i$ -coordinate of  $P$  was bounded by  $\varepsilon$ , the distance between  $P$  and  $P'$  is at most  $r\varepsilon$ , therefore  $P$  can be seen to have distance at most  $r\varepsilon < 2^{-2^{L+5}}$  to  $V(\Psi)$ . Furthermore,  $P$  is also contained in  $V(\Phi'_\infty)$ . This implies that  $V(\Phi'_\infty)$  has distance smaller than  $2^{-2^{L+5}}$  to  $V(\Psi)$ . This contradicts the result from applying Corollary 3.3.

We conclude that indeed  $V(\Phi) = \emptyset$  implies  $V(\Phi_\varepsilon) = \emptyset$ . This completes the proof of the lemma. ■

## 3.2 Almost Square Explicit Equality Constraints

Using Lemma 3.2, we are able to prove that an explicit version CE is also  $\exists\mathbb{R}$ -complete, with some additional assumptions. Note that this subsection is technically not needed for the proof of Theorem 1.11 and Theorem 1.13. We will prove a similar lemma also for the inequality case. And the inequality case can be used to also prove the equality case. Yet, we believe that seeing the proof first for the equality case makes it much easier to read Section 3.4.

**DEFINITION 3.4 (CE-EXPL).** Let  $U \subseteq \mathbb{R}$  and let  $f: U \rightarrow \mathbb{R}$  be a function. We define the CE-EXPL problem to be the CE problem corresponding to the function  $f^*: U \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f^*(x, y) = y - f(x)$ .

Note that for this definition of  $f^*$ , we have

$$\text{EqualZero}(f^*) = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x \notin U \vee y = f(x)\}.$$

In particular, this means that if we know that a variable  $x$  is forced to lie in  $U$ , then  $(x, y) \in \text{EqualZero}(f^*)$  will exactly imply that  $y = f(x)$ . In what follows, we will ensure we are in the case where all variables are contained in  $U$ , so this enables us to enforce constraints of the form  $y = f(x)$  on these variables, while also implying that the constructed instance is domain adherent.

The goal of this section is to prove the following result:

**LEMMA 3.5.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f: U \rightarrow \mathbb{R}$  be a function such that  $|f(x) - x^2| \leq \frac{1}{10}|x|^3$  for all  $x \in U \subseteq \mathbb{R}$ . Let  $T$  be strictly bounded from above by  $\frac{1}{4}$  and nicely computable. Furthermore, assume that the interval  $[-T(n), T(n)]$  is contained in  $U$  for all  $n$ . In this setting, CE-EXPL is  $\exists\mathbb{R}$ -hard, even when only considering instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

The reason that we impose these specific constraints on  $f$ , which enforce  $f$  to be very similar to squaring, is that the proof will use  $\exists\mathbb{R}$ -hardness of a problem involving a squaring constraint. Furthermore, this specific case can be generalized to more general functions  $f$ .

**PROOF.** Before giving the details of the construction, we will first give an overview of the used approach. The idea of this proof is to start with an instance of ETR-SQUARE, and convert

this into a CE-EXPL instance by using  $f$  to approximate squaring. In order to ensure that  $f$  approximates squaring close enough, the whole instance is scaled by some small factor  $\varepsilon$ , so every variable  $x$  is replaced by a variable representing  $\varepsilon x$  instead.

The linear constraints and inequalities are easy to rewrite in terms of  $\varepsilon x$ , for example a constraint of the form  $x + y = z$  can be rewritten to  $\varepsilon x + \varepsilon y = \varepsilon z$ .

Handling a squaring constraint  $y = x^2$  is a bit more complicated. The first step is to rewrite this to a constraint involving  $\varepsilon x$  and  $\varepsilon y$ , we get  $\varepsilon \cdot \varepsilon y = (\varepsilon x)^2$ . However, in the CE-EXPL problem there is no easy way to simulate the multiplication on the left-hand side of this equation. To solve this, we rewrite this equation to only use sums and differences of squares:

$$\varepsilon^2 + 2(\varepsilon x)^2 + (\varepsilon y)^2 - (\varepsilon + \varepsilon y)^2 = 0.$$

To simplify notation a bit, we will denote  $t_1 = \varepsilon$ ,  $t_2 = \varepsilon x$ ,  $t_3 = \varepsilon y$  and  $t_4 = \varepsilon + \varepsilon y$ . Using this notation the equation becomes  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$ . This is still not something we can directly enforce in a CE formula. However, by applying the function  $f$ , squaring can be approximated. Furthermore, Lemma 3.2 on a high level implies that such an approximation is enough to guarantee the existence of a solution to the original equations. This is why in the CE formulation we enforce

$$f(t_1) + 2f(t_2) + f(t_3) - f(t_4) = O(\varepsilon^3). \quad (3)$$

Enforcing the  $= O(\varepsilon^3)$  presents another problem: we cannot easily compute  $\varepsilon^3$ . To counter this, we instead bound the left-hand side of the equation in absolute value by  $2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon))$ , which is approximately equal to  $4\varepsilon^3$  (note that in the case that  $f(x) = x^2$  for all  $x$ , this expression would actually be equal to  $4\varepsilon^3 + 2\varepsilon^4$ ). The details of this reduction and a proof of its correctness will be worked out in the remainder of this proof.

**Reduction.** Let  $\Phi$  be an ETR-SQUARE formula. We will now construct a CE formula  $\Psi$  such that  $V(\Phi) \neq \emptyset$  if and only if  $V(\Psi) \neq \emptyset$ . The value of  $\delta$  will be defined at the end of the construction as  $T(n)$ , with  $n$  the final number of variables. Since the construction itself does not depend on the exact value of  $\delta$ , this does not cause any issues. The only important property is that  $\delta < \frac{1}{4}$ .

Let  $M$  be the constant obtained by applying Lemma 3.2 to  $\Phi$ , and let  $L$  be the smallest positive integer such that  $2^{-2^L} \leq \frac{1}{100} \cdot 2^{-2^M}$  and  $L \geq 3$ . We start by introducing variables  $[\delta_i]$  for  $0 \leq i \leq L$ . The variable  $[\delta_0]$  satisfies the constraint  $[\delta_0] = \delta$ , and for each  $1 \leq i \leq L$  we add a constraint

$$[\delta_i] = f([\delta_{i-1}]).$$

Denote  $[\varepsilon] = [\delta_L]$ . The idea behind these definitions is to simulate repeated squaring, as we will see later they force the value of  $[\varepsilon]$  to be in the interval  $(0, 2^{-2^L}]$ .



Next, we introduce a new variable  $\llbracket \approx 2\varepsilon^3 \rrbracket$  together with a (constant) number of constraints and auxiliary variables that enforce

$$\llbracket \approx 2\varepsilon^3 \rrbracket = f(\llbracket \varepsilon \rrbracket + f(\llbracket \varepsilon \rrbracket)) - f(\llbracket \varepsilon \rrbracket).$$

This can be done explicitly by introducing auxiliary variables  $\llbracket f(\varepsilon) \rrbracket$ ,  $\llbracket \varepsilon + f(\varepsilon) \rrbracket$  and  $\llbracket f(\varepsilon + f(\varepsilon)) \rrbracket$  and adding the following constraints:

$$\begin{aligned} \llbracket f(\varepsilon) \rrbracket &= f(\llbracket \varepsilon \rrbracket) \\ \llbracket \varepsilon + f(\varepsilon) \rrbracket &= \llbracket \varepsilon \rrbracket + \llbracket f(\varepsilon) \rrbracket \\ \llbracket f(\varepsilon + f(\varepsilon)) \rrbracket &= f(\llbracket \varepsilon + f(\varepsilon) \rrbracket) \\ \llbracket f(\varepsilon + f(\varepsilon)) \rrbracket &= \llbracket \approx 2\varepsilon^3 \rrbracket + \llbracket f(\varepsilon) \rrbracket. \end{aligned}$$

In the rest of this proof, and in future proofs of this paper, we will not give such explicit constraints anymore. The variable  $\llbracket \approx 2\varepsilon^3 \rrbracket$  will be used to bound the error on the constraints replacing squaring constraints, as mentioned in the overview of this proof. Stated differently, it replaces the “ $= O(\varepsilon^3)$ ” part of Equation (3).

Now, for each variable  $x$  of  $\Phi$ , we add a variable  $\llbracket \varepsilon x \rrbracket$  to  $\Psi$ , together with some constraints which enforce that  $-\llbracket \varepsilon \rrbracket \leq \llbracket \varepsilon x \rrbracket \leq \llbracket \varepsilon \rrbracket$ . Furthermore each constraint  $x + y = z$  is replaced by  $\llbracket \varepsilon x \rrbracket + \llbracket \varepsilon y \rrbracket = \llbracket \varepsilon z \rrbracket$ , each constraint  $x \geq 0$  is replaced by  $\llbracket \varepsilon x \rrbracket \geq 0$  and each constraint  $x = 1$  is replaced by  $\llbracket \varepsilon x \rrbracket = \llbracket \varepsilon \rrbracket$ .

For each constraint  $y = x^2$ , we build Equation (3) as in the overview. To do this, we first introduce variables  $\llbracket t_1 \rrbracket$ ,  $\llbracket t_2 \rrbracket$ ,  $\llbracket t_3 \rrbracket$  and  $\llbracket t_4 \rrbracket$  satisfying

$$\begin{aligned} \llbracket t_1 \rrbracket &= \llbracket \varepsilon \rrbracket \\ \llbracket t_2 \rrbracket &= \llbracket \varepsilon x \rrbracket \\ \llbracket t_3 \rrbracket &= \llbracket \varepsilon y \rrbracket \\ \llbracket t_4 \rrbracket &= \llbracket \varepsilon \rrbracket + \llbracket \varepsilon y \rrbracket. \end{aligned}$$

(Note that, even though  $x$  and  $y$  are suppressed in the notation here, the variables  $\llbracket t_1 \rrbracket$ ,  $\llbracket t_2 \rrbracket$ ,  $\llbracket t_3 \rrbracket$  and  $\llbracket t_4 \rrbracket$  should actually be distinct variables for each constraint  $y = x^2$ .) Next we introduce a new variable  $\llbracket \eta_{x,y} \rrbracket$  representing the left-hand side of Equation (3):

$$\llbracket \eta_{x,y} \rrbracket = f(\llbracket t_1 \rrbracket) + 2f(\llbracket t_2 \rrbracket) + f(\llbracket t_3 \rrbracket) - f(\llbracket t_4 \rrbracket).$$

The next step is to bound this variable, for this we add constraints which enforce

$$\begin{aligned} \llbracket \eta_{x,y} \rrbracket &\geq -2 \llbracket \approx 2\varepsilon^3 \rrbracket \quad \text{and} \\ \llbracket \eta_{x,y} \rrbracket &\leq 2 \llbracket \approx 2\varepsilon^3 \rrbracket. \end{aligned}$$

This completes the construction of  $\Psi$ . Now we can take  $\delta$  to equal  $T(n)$ , where  $n$  is the number of variables in  $\Psi$ . Note that in this construction, every constraint of  $\Phi$  is replaced by a constant

number of constraints in  $\Psi$ , and therefore we have  $|\Psi| = O(|\Phi|)$ . In particular this reduction can be executed in linear time.

**Calculations.** To show the validity of the reduction, we first perform some side calculations. We define  $\delta_0 = \delta$ , for  $1 \leq i \leq L$  we take  $\delta_i = f(\delta_{i-1})$ , and we take  $\varepsilon = \delta_L$ . We deduce the following facts:

$$|f(x) - x^2| \leq \frac{1}{10}|x|^3 \quad \text{for } x \in [-\delta, \delta] \setminus \{0\} \quad (4)$$

$$0 < f(x) \leq 2x^2 \quad \text{for } x \in [-\delta, \delta] \setminus \{0\} \quad (5)$$

$$\varepsilon \leq \frac{1}{100} \min(2^{-2^M}, \delta) \quad (6)$$

$$f(\varepsilon) < \varepsilon \quad (7)$$

$$f(\varepsilon + f(\varepsilon)) - f(\varepsilon) \in [\varepsilon^3, 3\varepsilon^3] \quad (8)$$

Inequality 4 is one of the assumptions and is repeated here just for clarity. Combining this with  $\delta \leq \frac{1}{4}$ , Inequality 5 follows.

Using induction with the fact that  $0 < f(x) \leq 2x^2$  for  $x \in [-\delta, \delta] \setminus \{0\}$ , it follows that  $0 < \delta_i \leq 2^{-2^{i-1}}$  for all  $i$ , so  $0 < \varepsilon \leq \frac{1}{2}2^{-2^L}$ .

Using the definition of  $L$ , we get that  $\varepsilon \leq \frac{1}{100}2^{-2^M}$ . Using that  $L \geq 3$  and  $\delta \leq 1/4$ , we get that  $\varepsilon \leq \delta^{2^3} = \delta^{16} \leq \frac{1}{100}\delta$ . Together this implies Inequality 6.

The fact  $f(\varepsilon) < \varepsilon$  now follows from Inequality 5 with the observation that  $\varepsilon < \min(\frac{1}{2}, \delta)$ , which follows from Inequality 6.

For deriving Inequality 8, we first rewrite by adding and subtracting some terms, and applying the triangle inequality:

$$\begin{aligned} |f(\varepsilon + f(\varepsilon)) - f(\varepsilon) - 2\varepsilon^3| &= |f(\varepsilon + f(\varepsilon)) - (\varepsilon + f(\varepsilon))^2 + \varepsilon^2 - f(\varepsilon) \\ &\quad + (f(\varepsilon) + \varepsilon^2)(f(\varepsilon) - \varepsilon^2) + \varepsilon^4 + 2\varepsilon(f(\varepsilon) - \varepsilon^2)| \\ &\leq |f(\varepsilon + f(\varepsilon)) - (\varepsilon + f(\varepsilon))^2| + |\varepsilon^2 - f(\varepsilon)| \\ &\quad + (f(\varepsilon) + \varepsilon^2)|f(\varepsilon) - \varepsilon^2| + \varepsilon^4 + 2\varepsilon|f(\varepsilon) - \varepsilon^2|. \end{aligned}$$

To this we apply Inequalities 4, 6 and 7 to obtain the desired bound, where in particular we use that Inequality 6 implies  $\varepsilon < \frac{1}{100}$ :

$$\begin{aligned} |f(\varepsilon + f(\varepsilon)) - f(\varepsilon) - 2\varepsilon^3| &\leq |f(\varepsilon + f(\varepsilon)) - (\varepsilon + f(\varepsilon))^2| + |\varepsilon^2 - f(\varepsilon)| \\ &\quad + (f(\varepsilon) + \varepsilon^2)|f(\varepsilon) - \varepsilon^2| + \varepsilon^4 + 2\varepsilon|f(\varepsilon) - \varepsilon^2| \\ &\leq \frac{1}{10}(\varepsilon + f(\varepsilon))^3 + \frac{1}{10}\varepsilon^3 + \frac{1}{10}(f(\varepsilon) + \varepsilon^2)\varepsilon^3 + \varepsilon^4 + \frac{1}{5}\varepsilon^4 \\ &\leq \frac{8}{10}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{1}{10}\varepsilon^3 \\ &\leq \varepsilon^3, \end{aligned}$$

so  $f(\varepsilon + f(\varepsilon)) - f(\varepsilon) \in [\varepsilon^3, 3\varepsilon^3]$ .

**$V(\Phi)$  nonempty implies  $V(\Psi)$  nonempty.** Now we can start to prove the validity of the reduction. First suppose that  $V(\Phi) \neq \emptyset$ , so there is some  $P \in V(\Phi)$ . It needs to be shown that also  $V(\Psi) \neq \emptyset$ , to do this we construct a point  $Q \in V(\Psi)$ . For a variable  $x$  of  $\Phi$ , we will use the notation  $x(P)$  for the value of this variable for the point  $P$ . A similar notation is used for  $Q$ . To define  $Q$ , we take  $\llbracket \varepsilon x \rrbracket (Q) = \varepsilon x(P)$  for all variables  $x$  of  $\Phi$ , and we enforce that  $Q$  satisfies all equality constraints of  $\Psi$ . This uniquely defines the value of  $Q$  in all other variables of  $\Psi$ . In particular, we get that

$$\begin{aligned} \llbracket \varepsilon \rrbracket (Q) &= \varepsilon \\ \llbracket \approx 2\varepsilon^3 \rrbracket (Q) &= f(\varepsilon + f(\varepsilon)) - f(\varepsilon) \\ \llbracket \eta_{x,y} \rrbracket (Q) &= f(\varepsilon) + 2f(\varepsilon x(P)) + f(\varepsilon y(P)) - f(\varepsilon + \varepsilon y(P)), \end{aligned}$$

where the last equality holds for all constraints  $y = x^2$  in  $\Phi$ .

It is left to show that  $Q$  also satisfies all inequalities of  $\Psi$ . There are three types of these inequalities. Firstly, we have inequalities which enforce  $|\llbracket \varepsilon x \rrbracket (Q)| \leq \llbracket \varepsilon \rrbracket (Q)$ . That these are satisfied for  $Q$  follows from the fact that  $|x(P)| \leq 1$  since  $\Phi$  is an ETR-SMALL formula. Secondly, for every inequality  $x \geq 0$  in  $\Phi$ , we get a corresponding inequality  $\llbracket \varepsilon x \rrbracket \geq 0$ , that this is satisfied follows by combining  $\llbracket \varepsilon x \rrbracket (Q) = \varepsilon x(P)$  and  $x(P) \geq 0$ .

Finally, for every constraint  $y = x^2$  in  $\Phi$  we get constraints enforcing  $|\llbracket \eta_{x,y} \rrbracket| \leq 2 \llbracket \approx 2\varepsilon^3 \rrbracket$ . To see that these are satisfied, first we shorten the notation a bit by writing  $t_1 = \llbracket t_1 \rrbracket (Q) = \varepsilon$ ,  $t_2 = \llbracket t_2 \rrbracket (Q) = \varepsilon x(P)$ ,  $t_3 = \llbracket t_3 \rrbracket (Q) = \varepsilon y(P)$  and  $t_4 = \llbracket t_4 \rrbracket (Q) = \varepsilon + \varepsilon y(P)$ . Now  $\llbracket \eta_{x,y} \rrbracket (Q)$  can be bounded, for this we first use the triangle inequality:

$$\begin{aligned} |\llbracket \eta_{x,y} \rrbracket (Q)| &= |f(t_1) + 2f(t_2) + f(t_3) - f(t_4)| \\ &= |f(t_1) - t_1^2 + 2(f(t_2) - t_2^2) + f(t_3) - t_3^2 - (f(t_4) - t_4^2) \\ &\quad + t_1^2 + 2t_2^2 + t_3^2 - t_4^2| \\ &\leq |f(t_1) - t_1^2| + 2|f(t_2) - t_2^2| + |f(t_3) - t_3^2| + |f(t_4) - t_4^2| \\ &\quad + |t_1^2 + 2t_2^2 + t_3^2 - t_4^2| \end{aligned}$$

Note that  $t_1, t_2, t_3$  and  $t_4$  were chosen in such a way to ensure that, given  $y(P) = x(P)^2$ , we have  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$ . Using this together with Inequality 4 and the facts that  $t_1, t_2$  and  $t_3$  are

bounded in absolute value by  $\varepsilon$ , and  $t_4$  is bounded in absolute value by  $2\varepsilon$ , we find

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q) | &\leq |f(t_1) - t_1^2| + 2|f(t_2) - t_2^2| + |f(t_3) - t_3^2| + |f(t_4) - t_4^2| \\ &\quad + |t_1^2 + 2t_2^2 + t_3^2 - t_4^2| \\ &\leq \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 + 0 \\ &\leq \left( \frac{1}{10} + \frac{1}{5} + \frac{1}{10} \right) \varepsilon^3 + \frac{1}{10} (2\varepsilon)^3 \\ &\leq \frac{6}{5} \varepsilon^3. \end{aligned}$$

Finally using Inequality 8 we derive

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q) | &\leq \frac{6}{5} \varepsilon^3 \\ &\leq 2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon)). \end{aligned}$$

This completes the proof that  $Q \in V(\Psi)$ , so  $V(\Psi) \neq \emptyset$ .

**$V(\Psi)$  nonempty implies  $V(\Phi)$  nonempty.** Next, suppose that there is some  $Q \in V(\Psi)$ . Now we want to show that  $|x(Q)| \leq \delta$  for all variables  $x$  in  $\Psi$ , and we want to prove that  $V(\Phi) \neq \emptyset$ . We start by bounding the coordinates. Note that the values  $\llbracket \delta_i \rrbracket (Q)$  can inductively be shown to be smaller than  $\delta$ . Here we use that  $\llbracket \delta_i \rrbracket (Q)$  being smaller than  $\delta$  implies that  $\llbracket \delta_i \rrbracket (Q) \in U$ , so the constraint  $\llbracket \delta_{i+1} \rrbracket = f(\llbracket \delta_i \rrbracket)$  actually enforces  $\llbracket \delta_{i+1} \rrbracket (Q) = f(\llbracket \delta_i \rrbracket (Q))$ . From this it follows that  $\llbracket \varepsilon \rrbracket (Q) = \varepsilon$ . Consequently, for every variable  $x$  in  $\Phi$ , it can be inferred that  $| \llbracket \varepsilon x \rrbracket (Q) | \leq | \llbracket \varepsilon \rrbracket (Q) | \leq \varepsilon$ . Using this, it can be shown that also all values of the auxiliary variables except for the  $\llbracket \delta_i \rrbracket$  are bounded by  $100\varepsilon \leq \delta$ . So this shows that  $Q$  is contained in  $[-\delta, \delta]^n$ , where  $n$  is the number of variables of  $\Psi$ . This also implies that  $Q$  is domain adherent, since  $[-\delta, \delta] \subseteq U$ .

Now we need to show that  $V(\Phi) \neq \emptyset$ . We apply Lemma 3.2 to show this. We will construct a point  $P$  within  $V(\Phi_{100\varepsilon})$ . This construction implies that  $V(\Phi)$  is non-empty, given that  $100\varepsilon \leq 2^{-2^M}$ . We define the point  $P$  by taking  $x(P) = \frac{\llbracket \varepsilon x \rrbracket (Q)}{\varepsilon}$  for all variables  $x$  of  $\Phi$ . It immediately follows that  $P$  satisfies all linear constraints and inequality constraints of  $\Phi$ , and it is only left to check that it satisfies the constraints  $|y - x^2| \leq 100\varepsilon$  of  $\Phi_{100\varepsilon}$ . To do this, first, we observe that, using Inequality 8,

$$\begin{aligned} | \llbracket \eta_{x,y} \rrbracket (Q) | &\leq 2 \llbracket \approx 2\varepsilon^3 \rrbracket (Q) \\ &= 2(f(\varepsilon + f(\varepsilon)) - f(\varepsilon)) \\ &\leq 6\varepsilon^3. \end{aligned}$$

Now we will try to bound  $|y(P) - x(P)^2|$ . First we again write  $t_1 = \llbracket t_1 \rrbracket (Q) = \varepsilon$ ,  $t_2 = \llbracket t_2 \rrbracket (Q) = \varepsilon x(P)$ ,  $t_3 = \llbracket t_3 \rrbracket (Q) = \varepsilon y(P)$  and  $t_4 = \llbracket t_4 \rrbracket (Q) = \varepsilon + \varepsilon y(P)$ . These choices were made such that

$$t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 2\varepsilon^2(x(P)^2 - y(P)),$$

so we see

$$2\varepsilon^2|y(P) - x(P)^2| = |t_1^2 + 2t_2^2 + t_3^2 - t_4^2|.$$

Next we apply the triangle inequality to get an expression to which we can apply Inequality 4 and the bound on  $|\llbracket \eta_{x,y} \rrbracket (Q)|$ :

$$\begin{aligned} 2\varepsilon^2|y(P) - x(P)^2| &= |t_1^2 + 2t_2^2 + t_3^2 - t_4^2| \\ &= |t_1^2 - f(t_1) + 2(t_2^2 - f(t_2)) + t_3^2 - f(t_3) - (t_4^2 - f(t_4)) \\ &\quad + f(t_1) + 2f(t_2) + f(t_3) - f(t_4)| \\ &\leq |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\ &\quad + |f(t_1) + 2f(t_2) + f(t_3) - f(t_4)| \\ &= |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\ &\quad + |\llbracket \eta_{x,y} \rrbracket (Q)| \end{aligned}$$

Applying Inequality 4 and the bound on  $|\llbracket \eta_{x,y} \rrbracket (Q)|$  yields

$$\begin{aligned} 2\varepsilon^2|y(P) - x(P)^2| &\leq |t_1^2 - f(t_1)| + 2|t_2^2 - f(t_2)| + |t_3^2 - f(t_3)| + |t_4^2 - f(t_4)| \\ &\quad + |\llbracket \eta_{x,y} \rrbracket (Q)| \\ &\leq \frac{1}{10}t_1^3 + \frac{1}{5}t_2^3 + \frac{1}{10}t_3^3 + \frac{1}{10}t_4^3 + 6\varepsilon^3 \end{aligned}$$

Finally we use that  $t_1, t_2$  and  $t_3$  are bounded in absolute value by  $\varepsilon$ , and that  $t_4$  is bounded by  $2\varepsilon$ :

$$\begin{aligned} 2\varepsilon^2|y(P) - x(P)^2| &\leq \frac{1}{10}t_1^3 + \frac{1}{5}t_2^3 + \frac{1}{10}t_3^3 + \frac{1}{10}t_4^3 + 6\varepsilon^3 \\ &\leq \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 + 6\varepsilon^3 \\ &< 200\varepsilon^3. \end{aligned}$$

So  $|y(P) - x(P)^2| \leq 100\varepsilon$ . This proves that  $P \in V(\Phi_{100\varepsilon})$ , and therefore  $V(\Phi) \neq \emptyset$ .

This finishes the proof of the validity of the reduction of ETR-SQUARE to CE-EXPL. We conclude that for the given  $f$ , the problem CE-EXPL is  $\exists\mathbb{R}$ -hard. ■

### 3.3 Almost Square Explicit Inequality Constraints

In this section, we will prove a number of hardness results about the explicit version of CCI. Before we can describe these results, we first need the following definition:

**DEFINITION 3.6 (CCI-EXPL).** Let  $U \subseteq \mathbb{R}$  and let  $f, g: U \rightarrow \mathbb{R}$  be two functions. Now we define the CCI-EXPL problem to be the CCI problem corresponding to the functions  $f^*, g^*: U \times \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f^*(x, y) = y - f(x)$  and  $g^*(x, y) = g(x) - y$ .

Note that, just as in Definition 3.4, the constraints  $\text{LargerZero}(f^*)$  and  $\text{LargerZero}(g^*)$  in this definition can be used to enforce  $y \geq f(x)$  and  $y \leq g(x)$  if we already know that  $y \in U$ .

We will prove that CCI-EXPL is  $\exists\mathbb{R}$ -hard in a large number of cases. In particular, we prove the following:

**COROLLARY 3.7.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions which are three times differentiable such that  $f(0) = g(0) = 0$  and  $f'(0), f''(0), g'(0), g''(0) \in \mathbb{Q}$  with  $f''(0), g''(0) > 0$ . Let  $T$  be bounded and nicely computable. In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

This corollary will be an important step towards proving Theorem 1.11. Before we can prove this corollary, we first prove another result which is similar to Lemma 3.5.

**LEMMA 3.8.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions such that  $|f(x) - x^2| \leq \frac{1}{10}|x|^3$  and  $|g(x) - x^2| \leq \frac{1}{10}|x|^3$  for all  $x \in U$ . Let  $T$  be strictly bounded from above by  $\frac{1}{4}$  and nicely computable. Furthermore, assume that the interval  $[-T(n), T(n)]$  is contained in  $U$  for all  $n$ . In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when only considering instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** The idea is to use almost the same construction as in Lemma 3.5, so we recommend the reader to first read the proof of this lemma. The first main difference is that some extra care needs to be taken when making the constraints for the  $[[\delta_i]]$  variables. Also, the squaring constraints need to be handled in a slightly different way. In order to do this, we replace the variables  $[[\eta_{x,y}]]$  by two new variables  $[[\eta_{x,y}^{\text{low}}]]$  and  $[[\eta_{x,y}^{\text{high}}]]$ , which impose a lower bound, respectively upper bound, on the value of  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2$ . Here  $t_1 = \varepsilon$ ,  $t_2 = \varepsilon x$ ,  $t_3 = \varepsilon y$  and  $t_4 = \varepsilon + \varepsilon y$ , as before.

**Reduction.** Let  $\Phi$  be an ETR-SQUARE formula. We will construct a CCI-EXPL formula  $\Psi$  such that  $V(\Phi) \neq \emptyset$  if and only if  $V(\Psi) \neq \emptyset$ . Again we will take  $\delta = T(n) < \frac{1}{4}$  at the end of the construction, where  $n$  is the final number of variables. Let  $M$  be the constant obtained by applying Lemma 3.2 to  $\Phi$ , and let  $L$  be a constant such that  $2^{-2^L} \leq \frac{1}{100} \cdot 2^{-2^M}$  and  $L \geq 3$ , just like in the proof of Lemma 3.5.

We again introduce  $[[\delta_i]]$  for  $0 \leq i \leq L$ , where the variable  $[[\delta_0]]$  should satisfy the constraint  $[[\delta_0]] = \delta$ . For each  $1 \leq i \leq L$  we now add constraints enforcing

$$\frac{1}{2}f([[ \delta_{i-1} ]]) \leq [[ \delta_i ]] \leq g([[ \delta_{i-1} ]]).$$

Denote  $[[\varepsilon]] = [[\delta_L]]$ . The constraints  $[[\delta_i]] \leq g([[ \delta_{i-1} ]])$  are there to enforce that  $[[\varepsilon]] \leq 2^{-2^L}$ , and the constraints  $\frac{1}{2}f([[ \delta_{i-1} ]]) \leq [[ \delta_i ]]$  are there to enforce that  $[[\varepsilon]] > 0$ .

We continue by defining variables  $\llbracket \leq g(\varepsilon) \rrbracket$  and  $\llbracket \lesssim 2\varepsilon^3 \rrbracket$  using constraints

$$\begin{aligned} \llbracket \leq g(\varepsilon) \rrbracket &\leq g(\llbracket \varepsilon \rrbracket), \\ \llbracket \leq g(\varepsilon) \rrbracket &\geq 0, \\ \llbracket \lesssim 2\varepsilon^3 \rrbracket &\leq g(\llbracket \varepsilon \rrbracket + \llbracket \leq g(\varepsilon) \rrbracket) - f(\llbracket \varepsilon \rrbracket), \\ \llbracket \lesssim 2\varepsilon^3 \rrbracket &\geq 0. \end{aligned}$$

This new variable  $\llbracket \lesssim 2\varepsilon^3 \rrbracket$  is a replacement for the variable  $\llbracket \approx 2\varepsilon^3 \rrbracket$  which occurred in the proof of Lemma 3.5. Later, we will show that  $\llbracket \lesssim 2\varepsilon^3 \rrbracket$  is upper bounded by  $3\varepsilon^3$ .

Next for each variable  $x$  of  $\Phi$ , we add a variable  $\llbracket \varepsilon x \rrbracket$  to  $\Psi$ , with constraints enforcing  $-\llbracket \varepsilon \rrbracket \leq \llbracket \varepsilon x \rrbracket \leq \llbracket \varepsilon \rrbracket$ . Constraints of type  $x + y = z$ , type  $x \geq 0$  or type  $x = 1$  are handled by replacing them by constraints  $\llbracket \varepsilon x \rrbracket + \llbracket \varepsilon y \rrbracket = \llbracket \varepsilon z \rrbracket$ ,  $\llbracket \varepsilon x \rrbracket \geq 0$  and  $\llbracket \varepsilon x \rrbracket = \llbracket \varepsilon \rrbracket$ , respectively.

For each constraint  $y = x^2$ , we introduce variables  $\llbracket t_1 \rrbracket$ ,  $\llbracket t_2 \rrbracket$ ,  $\llbracket t_3 \rrbracket$  and  $\llbracket t_4 \rrbracket$  with constraints

$$\begin{aligned} \llbracket t_1 \rrbracket &= \llbracket \varepsilon \rrbracket, \\ \llbracket t_2 \rrbracket &= \llbracket \varepsilon x \rrbracket, \\ \llbracket t_3 \rrbracket &= \llbracket \varepsilon y \rrbracket, \\ \llbracket t_4 \rrbracket &= \llbracket \varepsilon \rrbracket + \llbracket \varepsilon y \rrbracket. \end{aligned}$$

Next we introduce two new variables:  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket$  and  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$ , together with constraints enforcing

$$\begin{aligned} \llbracket \eta_{x,y}^{\text{low}} \rrbracket &\leq g(\llbracket t_1 \rrbracket) + 2g(\llbracket t_2 \rrbracket) + g(\llbracket t_3 \rrbracket) - f(\llbracket t_4 \rrbracket), \\ \llbracket \eta_{x,y}^{\text{high}} \rrbracket &\geq f(\llbracket t_1 \rrbracket) + 2f(\llbracket t_2 \rrbracket) + f(\llbracket t_3 \rrbracket) - g(\llbracket t_4 \rrbracket), \\ \llbracket \eta_{x,y}^{\text{low}} \rrbracket &\geq -2 \llbracket \lesssim 2\varepsilon^3 \rrbracket, \\ \llbracket \eta_{x,y}^{\text{high}} \rrbracket &\leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket. \end{aligned}$$

Note that the variables  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket$  and  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$  are not completely necessary, and that it is also possible to use direct constraints

$$\begin{aligned} -2 \llbracket \lesssim 2\varepsilon^3 \rrbracket &\leq g(\llbracket t_1 \rrbracket) + 2g(\llbracket t_2 \rrbracket) + g(\llbracket t_3 \rrbracket) - f(\llbracket t_4 \rrbracket), \\ 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket &\geq f(\llbracket t_1 \rrbracket) + 2f(\llbracket t_2 \rrbracket) + f(\llbracket t_3 \rrbracket) - g(\llbracket t_4 \rrbracket) \end{aligned}$$

instead. The two variables  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket$  and  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$  are included since these slightly simplify the notation when proving correctness of this construction later on. This completes the construction of  $\Psi$ , which can be performed in linear time. We finish by choosing  $\delta = T(n)$  with  $n$  being the number of variables in  $\Psi$ . This completes the CCI-EXPL instance.

**Calculations.** Let  $\varepsilon$  be any real number such that there exist reals  $\delta_i$  for  $0 \leq i \leq L$  satisfying  $\delta_0 = \delta$ ,  $\delta_L = \varepsilon$  and  $\frac{1}{2}f(\delta_{i-1}) \leq \delta_i \leq g(\delta_{i-1})$  for all  $1 \leq i \leq L$ . Now we have the following facts

(these facts hold in particular if  $\varepsilon = \llbracket \varepsilon \rrbracket (Q)$  for some  $Q \in V(\Psi)$ ):

$$|f(x) - x^2| \leq \frac{1}{10}|x|^3 \quad \text{for } x \in [-\delta, \delta], \quad (9)$$

$$|g(x) - x^2| \leq \frac{1}{10}|x|^3 \quad \text{for } x \in [-\delta, \delta], \quad (10)$$

$$\frac{1}{2}f(x) \leq g(x) \quad \text{for } x \in [-\delta, \delta], \quad (11)$$

$$\varepsilon \leq \frac{1}{100} \min(2^{-2M}, \delta), \quad (12)$$

$$f(\varepsilon) < \varepsilon, \quad g(\varepsilon) < \varepsilon, \quad (13)$$

$$\llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) \leq 3 \llbracket \varepsilon \rrbracket (Q)^3 \quad \text{for } Q \in V(\Psi), \quad (14)$$

$$g(\varepsilon + g(\varepsilon)) - f(\varepsilon) \geq \varepsilon^3. \quad (15)$$

Inequalities 9 and 10 are assumptions from the statement of the lemma. Inequality 11 follows from this, together with the fact that  $|x|$  is bounded by  $\frac{1}{4}$ :

$$\frac{1}{2}f(x) \leq \frac{1}{2}x^2 + \frac{1}{20}|x|^3 \leq x^2 - \frac{1}{10}|x|^3 \leq g(x).$$

Inequality 12 can be derived in the same way as Inequality 6 from Lemma 3.5, and now Inequality 13 follows from this with Inequalities 9 and 10.

In order to derive Inequality 14, we use the definition of the variable  $\llbracket \lesssim 2\varepsilon^3 \rrbracket$  and apply Inequalities 9 and 10 to this (here we take  $\varepsilon = \llbracket \varepsilon \rrbracket (Q)$  to simplify the notation a bit):

$$\begin{aligned} \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) &\leq g(\varepsilon + \llbracket \leq g(\varepsilon) \rrbracket (Q)) - f(\varepsilon) \\ &\leq (\varepsilon + \llbracket \leq g(\varepsilon) \rrbracket (Q))^2 + \frac{1}{10}(\varepsilon + \llbracket \leq g(\varepsilon) \rrbracket (Q))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3. \end{aligned}$$

Combining this with the constraint  $\llbracket \leq g(\varepsilon) \rrbracket \leq g(\llbracket \varepsilon \rrbracket)$  and Inequalities 10 and 13, we get

$$\begin{aligned} \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) &\leq (\varepsilon + \llbracket \leq g(\varepsilon) \rrbracket (Q))^2 + \frac{1}{10}(\varepsilon + \llbracket \leq g(\varepsilon) \rrbracket (Q))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3 \\ &\leq (\varepsilon + g(\varepsilon))^2 + \frac{1}{10}(\varepsilon + g(\varepsilon))^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3 \\ &\leq \left(\varepsilon + \varepsilon^2 + \frac{1}{10}\varepsilon^3\right)^2 + \frac{1}{10}(\varepsilon + \varepsilon)^3 - \varepsilon^2 + \frac{1}{10}\varepsilon^3 \\ &= \frac{29}{10}\varepsilon^3 + \frac{6}{5}\varepsilon^4 + \frac{1}{5}\varepsilon^5 + \frac{1}{100}\varepsilon^6. \end{aligned}$$

Combining this with  $\varepsilon \leq \frac{1}{100}$  (which follows from Inequality 12) yields that  $\llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) \leq 3\varepsilon^3$ , as we wanted.



Finally Inequality 15 follows from Inequalities 9, 10 and 13 in the following manner:

$$\begin{aligned}
g(\varepsilon + g(\varepsilon)) - f(\varepsilon) &\geq (\varepsilon + g(\varepsilon))^2 - \frac{1}{10}(\varepsilon + g(\varepsilon))^3 - \varepsilon^2 - \frac{1}{10}\varepsilon^3 \\
&\geq \left(\varepsilon + \varepsilon^2 - \frac{1}{10}\varepsilon^3\right)^2 - \frac{1}{10}(\varepsilon + \varepsilon)^3 - \varepsilon^2 - \frac{1}{10}\varepsilon^3 \\
&= \frac{11}{10}\varepsilon^3 + \frac{4}{5}\varepsilon^4 - \frac{1}{5}\varepsilon^5 + \frac{1}{100}\varepsilon^6 \\
&\geq \varepsilon^3.
\end{aligned}$$

**$V(\Phi)$  nonempty implies  $V(\Psi)$  nonempty.** Suppose that  $V(\Phi) \neq \emptyset$ , and therefore, there exists some  $P \in V(\Phi)$ . Our goal is to demonstrate that  $V(\Psi) \neq \emptyset$ . To achieve this, we construct a point  $Q$  that lies within  $V(\Psi)$ . We start by taking  $\llbracket \delta_0 \rrbracket(Q) = \delta$  and  $\llbracket \delta_i \rrbracket(Q) = g(\llbracket \delta_{i-1} \rrbracket)$  for all  $1 \leq i \leq L$ . By Inequality 11, this definition satisfies all constraints on the  $\llbracket \delta_i \rrbracket$ . Denote  $\varepsilon = \llbracket \varepsilon \rrbracket(Q) = \llbracket \delta_L \rrbracket(Q)$ . Next we take  $\llbracket \leq g(\varepsilon) \rrbracket(Q) = g(\varepsilon)$  and  $\llbracket \lesssim 2\varepsilon^3 \rrbracket(Q) = g(\varepsilon + g(\varepsilon)) - f(\varepsilon)$ , so by Inequality 15 we know that  $\llbracket \lesssim 2\varepsilon^3 \rrbracket(Q) \geq \varepsilon^3$ .

For all variables  $x$  of  $\Phi$ , we take  $\llbracket \varepsilon x \rrbracket(Q) = \varepsilon x(P)$ . Since  $V(P) \subseteq [-1, 1]^n$ , it follows that all inequalities of the form  $-\llbracket \varepsilon \rrbracket(Q) \leq \llbracket \varepsilon x \rrbracket(Q) \leq \llbracket \varepsilon \rrbracket(Q)$  are satisfied in this way. Also for every constraint from  $\Phi$  of one of the forms  $x + y = z$ ,  $x \geq 0$  or  $x = 1$ , the corresponding constraint in  $\Psi$  is clearly satisfied.

Next we consider a squaring constraint  $y = x^2$  from  $\Phi$ , for each such constraint we take

$$\begin{aligned}
\llbracket t_1 \rrbracket(Q) &= \varepsilon, \\
\llbracket t_2 \rrbracket(Q) &= \llbracket \varepsilon x \rrbracket(Q), \\
\llbracket t_3 \rrbracket(Q) &= \llbracket \varepsilon y \rrbracket(Q), \\
\llbracket t_4 \rrbracket(Q) &= \varepsilon + \llbracket \varepsilon y \rrbracket(Q), \\
\llbracket \eta_{x,y}^{\text{low}} \rrbracket(Q) &= g(\llbracket t_1 \rrbracket(Q)) + 2g(\llbracket t_2 \rrbracket(Q)) + g(\llbracket t_3 \rrbracket(Q)) - f(\llbracket t_4 \rrbracket(Q)), \\
\llbracket \eta_{x,y}^{\text{high}} \rrbracket(Q) &= f(\llbracket t_1 \rrbracket(Q)) + 2f(\llbracket t_2 \rrbracket(Q)) + f(\llbracket t_3 \rrbracket(Q)) - g(\llbracket t_4 \rrbracket(Q)).
\end{aligned}$$

Using these definitions, the only constraints for which we still need to check whether  $Q$  satisfies them, are the constraints of the form  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket \geq -2 \llbracket \lesssim 2\varepsilon^3 \rrbracket$  and  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket \leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket$ .

We start by checking the first of these constraints. Denote  $t_1 = \llbracket t_1 \rrbracket(Q)$ ,  $t_2 = \llbracket t_2 \rrbracket(Q)$ ,  $t_3 = \llbracket t_3 \rrbracket(Q)$  and  $t_4 = \llbracket t_4 \rrbracket(Q)$ . Now we can apply Inequalities 9 and 10 to the definition of  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket$ :

$$\begin{aligned}
\llbracket \eta_{x,y}^{\text{low}} \rrbracket(Q) &= g(t_1) + 2g(t_2) + g(t_3) - f(t_4) \\
&\geq t_1^2 - \frac{1}{10}|t_1|^3 + 2t_2^2 - \frac{1}{5}|t_2|^3 + t_3^2 - \frac{1}{10}|t_3|^3 - t_4^2 - \frac{1}{10}|t_4|^3 \\
&= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 - \left(\frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3\right).
\end{aligned}$$

Since  $t_1, \dots, t_4$  were chosen such that  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = \varepsilon^2 x(P)^2 - \varepsilon^2 y(P)$ , and by the fact that  $y(P) = x(P)^2$ , it follows that  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$ . Furthermore,  $t_1, t_2$  and  $t_3$  are all bounded by  $\varepsilon$  in absolute value, while  $|t_4|$  is bounded by  $2\varepsilon$ . This yields

$$\begin{aligned} \llbracket \eta_{x,y}^{\text{low}} \rrbracket (Q) &\geq t_1^2 + 2t_2^2 + t_3^2 - t_4^2 - \left( \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \right) \\ &\geq 0 - \left( \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 \right) \\ &\geq -2\varepsilon^3. \end{aligned}$$

Combining this with  $\llbracket \lesssim 2\varepsilon^3 \rrbracket (Q) \geq \varepsilon^3$ , we find  $\llbracket \eta_{x,y}^{\text{low}} \rrbracket (Q) \geq -2 \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q)$ .

Next we consider the constraint  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket \leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket$ . We apply Inequalities 9 and 10 to the definition of  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$ :

$$\begin{aligned} \llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) &= f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\ &\leq t_1^2 + \frac{1}{10}|t_1|^3 + 2t_2^2 + \frac{1}{5}|t_2|^3 + t_3^2 + \frac{1}{10}|t_3|^3 - t_4^2 + \frac{1}{10}|t_4|^3 \\ &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3. \end{aligned}$$

Here we can apply that  $t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 0$ , and that  $|t_1|, |t_2|$  and  $|t_3|$  are bounded by  $\varepsilon$  and  $|t_4|$  is bounded by  $2\varepsilon$  to get

$$\begin{aligned} \llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) &\leq t_1^2 + 2t_2^2 + t_3^2 - t_4^2 + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \\ &\leq 0 + \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 \\ &\leq 2\varepsilon^3. \end{aligned}$$

So we get that  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket (Q) \leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket (Q)$ .

We conclude that  $Q$  satisfies all constraints from  $\Psi$ , and therefore  $Q \in V(\Psi)$ . This proves that  $V(\Psi) \neq \emptyset$ .

**$V(\Psi)$  nonempty implies  $V(\Phi)$  nonempty.** Next, let  $Q \in V(\Psi)$ . Just as in the proof of Lemma 3.5, we want to show that  $|x(Q)| \leq \delta$  for all variables  $x$  of  $\Psi$ , and we want to prove that  $V(\Phi) \neq \emptyset$ . Bounding the coordinates and showing that  $Q$  is domain adherent goes in exactly the same way as in Lemma 3.5.

In order to demonstrate that  $V(\Phi) \neq \emptyset$ , we once more apply Lemma 3.2. We construct a point  $P$  in  $V(\Phi_{100\varepsilon})$ , where we again denote  $\varepsilon = \llbracket \varepsilon \rrbracket (Q)$ . This would imply  $V(\Phi) \neq \emptyset$  since  $100\varepsilon \leq 2^{-2^M}$ . We take  $x(P) = \frac{\llbracket \varepsilon x \rrbracket (Q)}{\varepsilon}$  for all variables  $x$  of  $\Phi$ . Now  $P$  satisfies all linear constraints and inequality constraints of  $\Phi$ , and it only remains to be checked that it satisfies the constraints  $|y - x^2| \leq 100\varepsilon$  of  $\Phi_{100\varepsilon}$ .

We start by proving that  $x(P)^2 - y(P) \leq 100\varepsilon$ . Denote  $t_1 = \llbracket t_1 \rrbracket(Q) = \varepsilon$ ,  $t_2 = \llbracket t_2 \rrbracket(Q) = \varepsilon x(P)$ ,  $t_3 = \llbracket t_3 \rrbracket(Q) = \varepsilon y(P)$  and  $t_4 = \llbracket t_4 \rrbracket(Q) = \varepsilon + \varepsilon y(P)$ . We have that

$$t_1^2 + 2t_2^2 + t_3^2 - t_4^2 = 2\varepsilon^2(x(P)^2 - y(P)).$$

Note that from Inequalities 9 and 10 it also follows that  $x^2 \leq f(x) + \frac{1}{10}|x|^3$  and  $x^2 \geq g(x) - \frac{1}{10}|x|^3$  for all  $x \in [-\delta, \delta]$ . Using this, we find

$$\begin{aligned} 2\varepsilon^2(x(P)^2 - y(P)) &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 \\ &\leq f(t_1) + \frac{1}{10}|t_1|^3 + 2f(t_2) + \frac{1}{5}|t_2|^3 \\ &\quad + f(t_3) + \frac{1}{10}|t_3|^3 - g(t_4) + \frac{1}{10}|t_4|^3 \\ &= f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\ &\quad + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3. \end{aligned}$$

To bound this, we use the variable  $\llbracket \eta_{x,y}^{\text{high}} \rrbracket$ , and the observation that  $t_1, t_2$  and  $t_3$  are bounded in absolute value by  $\varepsilon$ , and  $|t_4|$  is bounded by  $2\varepsilon$ :

$$\begin{aligned} 2\varepsilon^2(x(P)^2 - y(P)) &\leq f(t_1) + 2f(t_2) + f(t_3) - g(t_4) \\ &\quad + \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \\ &\leq \llbracket \eta_{x,y}^{\text{high}} \rrbracket(Q) + \frac{1}{10}\varepsilon^3 + \frac{1}{5}\varepsilon^3 + \frac{1}{10}\varepsilon^3 + \frac{8}{10}\varepsilon^3 \\ &\leq 2 \llbracket \lesssim 2\varepsilon^3 \rrbracket(Q) + 2\varepsilon^3. \end{aligned}$$

Here we can apply Inequality 14 to find

$$2\varepsilon^2(x(P)^2 - y(P)) \leq 8\varepsilon^3 < 200\varepsilon^3.$$

This implies  $x(P)^2 - y(P) \leq 100\varepsilon$ , as we wanted.

The proof that  $x(P)^2 - y(P) \geq -100\varepsilon$  works in a similar manner. Leaving out some intermediate steps, it looks as follows:

$$\begin{aligned} 2\varepsilon^2(x(P)^2 - y(P)) &= t_1^2 + 2t_2^2 + t_3^2 - t_4^2 \\ &\geq g(t_1) + 2g(t_2) + g(t_3) - f(t_4) \\ &\quad - \left( \frac{1}{10}|t_1|^3 + \frac{1}{5}|t_2|^3 + \frac{1}{10}|t_3|^3 + \frac{1}{10}|t_4|^3 \right) \\ &\geq \llbracket \eta_{x,y}^{\text{low}} \rrbracket(Q) - 2\varepsilon^3 \\ &\geq -8\varepsilon^3 > -200\varepsilon^3, \end{aligned}$$

and therefore  $x(P)^2 - y(P) \geq -100\varepsilon$ . This implies that  $P \in V(\Phi_{100\varepsilon})$ , and therefore  $V(\Phi) \neq \emptyset$ .

This completes the proof of the validity of the reduction of ETR-SQUARE to CCI-EXPL. So for  $f$  and  $g$  satisfying the conditions from the lemma, the problem CCI-EXPL is  $\exists\mathbb{R}$ -hard. ■

Now that hardness of this restricted version of CCI-EXPL is proven, this result can be generalized in small steps until finally Theorem 1.13 is proven.

Before we do this, we first note that in any CCI-EXPL formula, constraints of the form  $x = q \cdot y$ , where  $x, y$  are variables and  $q$  is a rational constant, can be enforced using a constant number of addition constraints and new variables. To illustrate this, we will discuss the case where  $q \in [0, 1]$  here. Other cases can be handled in a similar manner. Assume that  $q = a/b$  for a positive integer  $b$  and an integer  $0 \leq a \leq b$ . Now we can introduce variables  $\llbracket \frac{i}{b}y \rrbracket$  for all  $0 \leq i \leq b$ , which satisfy constraints

$$\begin{aligned} \llbracket \frac{0}{b}y \rrbracket &= \llbracket \frac{0}{b}y \rrbracket + \llbracket \frac{0}{b}y \rrbracket, \\ \llbracket \frac{i+1}{b}y \rrbracket &= \llbracket \frac{i}{b}y \rrbracket + \llbracket \frac{1}{b}y \rrbracket && \text{for } 0 \leq i < b, \\ \llbracket \frac{b}{b}y \rrbracket &= \llbracket \frac{0}{b}y \rrbracket + y, \\ \llbracket \frac{a}{b}y \rrbracket &= \llbracket \frac{0}{b}y \rrbracket + x. \end{aligned}$$

This exactly enforces that  $x = \frac{a}{b} \cdot y$ .

The first step in working from Lemma 3.8 to Theorem 1.13 is to get rid of the constraint that  $T(n)$  has to be bounded by  $\frac{1}{4}$ .

**LEMMA 3.9.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions such that  $|f(x) - x^2| \leq \frac{1}{10}|x|^3$  and  $|g(x) - x^2| \leq \frac{1}{10}|x|^3$  for all  $x \in U$ . Let  $T$  be bounded and nicely computable. In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** Let  $c_1$  be some rational constant such that  $0 < c_1 < \frac{1}{4}$  and  $[-c_1, c_1] \subseteq U$ . We will give a self-reduction from instances with  $\delta = T^*(n)$  to instances with  $\delta = T(n)$ , where  $T^*$  is some nicely computable function bounded by  $c_1$  which is yet to be determined.

Let  $c_2$  be a rational constant that strictly bounds  $T$  from above, and denote  $c = \frac{c_2}{c_1}$ . Without loss of generality we may assume that  $c_2 \geq \frac{1}{4}$ , so in particular  $c \geq 1$ . Let  $(\delta, \Phi)$  be some instance of CCI-EXPL with  $n$  variables, and  $\delta = T^*(n)$ . We will build an equisatisfiable instance  $(\delta', \Psi)$  with  $m$  variables and fix  $T^*$ , such that  $\delta' = c\delta = T(m)$ . For this, we add to  $\Phi$  extra variables  $\llbracket \delta \rrbracket$  and  $\llbracket \delta' \rrbracket$ , together with constraints and auxiliary variables enforcing  $\llbracket \delta' \rrbracket = \delta'$  and  $\llbracket \delta' \rrbracket = c \llbracket \delta \rrbracket$ . Next we replace every constraint of the form  $x = \delta$  by a constraint of the form  $x = \llbracket \delta \rrbracket$ . This gives us the formula  $\Psi$ . Note that the solutions to  $\Phi$  directly correspond to solutions of  $\Psi$ . The fact that  $\delta' \geq \delta$  and the promises on  $\Psi$  imply that all solutions of  $\Psi$  are contained in  $[-\delta', \delta']$ . Furthermore, the fact that  $\Phi$  is domain adherent implies that also  $\Psi$  is domain adherent.

Note that the number of variables in  $\Psi$  is exactly  $n$  plus some constant  $d$  which only depends on the function  $T$ . Therefore we can take  $T^*(n) = \frac{1}{c}T(n+d)$  and  $\delta' = T(n+d)$ , this ensures that in the preceding construction we have  $\delta' = c \cdot \delta$ , and therefore  $\Phi$  and  $\Psi$  are indeed

equisatisfiable. Furthermore, since  $T$  is bounded from above by  $c_2$ , it follows that  $T^*$  is bounded from above by  $c_1$ . In particular, it follows that  $[-T^*(n), T^*(n)] \subseteq U$  for all values of  $n$ . ■

Now the next step when working towards Theorem 1.13, is to slightly relax the constraints on  $f$  and  $g$ , by allowing the difference with squaring to be any  $O(x^3)$  function, instead of just functions bounded by  $\frac{1}{10}|x|^3$  in absolute value.

**LEMMA 3.10.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions such that  $f(x) = x^2 + O(x^3)$  and  $g(x) = x^2 + O(x^3)$  as  $x \rightarrow 0$ . Let  $T$  be bounded and nicely computable. In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when only considering instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** Let  $c$  be a constant such that  $|f(x) - x^2| \leq c|x|^3$  and  $|g(x) - x^2| \leq c|x|^3$  for all  $x \in U^*$  where  $U^* \subseteq U$  is a neighborhood of 0. Now let  $N$  be a positive integer larger than  $10c$ . This implies that for all  $x \in U^*$

$$\begin{aligned} |N^2 f(x/N) - x^2| &\leq \frac{1}{10}|x|^3 \text{ and} \\ |N^2 g(x/N) - x^2| &\leq \frac{1}{10}|x|^3. \end{aligned}$$

If we define  $f^*$  and  $g^*$  on the domain  $U^*$  by  $f^*(x) = N^2 f(x/N)$  and  $g^*(x) = N^2 g(x/N)$ , then using Lemma 3.9, we get that the problem CCI-EXPL is  $\exists\mathbb{R}$ -hard for  $f^*$  and  $g^*$ . For the rest of the proof of this lemma, we will denote this specific CCI-EXPL version using  $f^*$  and  $g^*$  by CCI-EXPL\*.

We give a reduction from CCI-EXPL\* to the CCI-EXPL version with  $f$  and  $g$ . Let  $(\delta, \Phi)$  be a CCI-EXPL\* instance. Now for every variable  $x$  in this instance, we add extra variables  $\llbracket x/N \rrbracket$ ,  $\llbracket x/N^2 \rrbracket$  and we add constraints enforcing

$$\begin{aligned} \llbracket \frac{x}{N} \rrbracket &= \frac{\llbracket x \rrbracket}{N}, \\ \llbracket \frac{x}{N^2} \rrbracket &= \frac{\llbracket x \rrbracket}{N^2}. \end{aligned}$$

Next we replace every constraint of the form  $y \geq f^*(x)$  (i.e.  $\text{LargerZero}(y - f^*(x))$ ) by a constraint  $\llbracket y/N^2 \rrbracket \geq f(\llbracket x/N \rrbracket)$  (i.e.  $\text{LargerZero}(\llbracket y/N^2 \rrbracket - f(\llbracket x/N \rrbracket))$ ). Similarly, we replace every constraint of the form  $y \leq g^*(x)$  (i.e.  $\text{LargerZero}(g^*(x) - y)$ ) by a constraint  $\llbracket y/N^2 \rrbracket \leq g(\llbracket x/N \rrbracket)$  (i.e.  $\text{LargerZero}(g(\llbracket x/N \rrbracket) - \llbracket y/N^2 \rrbracket)$ ).

This results in a CCI-EXPL formula  $\Psi$ . Note that any solution of  $\Phi$  is domain adherent and contained in  $[-\delta, \delta]^n$ , and that the domain  $U^*$  of  $f^*$  and  $g^*$  is a subset of  $U$ . From this, it follows that any solution of  $\Phi$  corresponds to a domain adherent solution of  $\Psi$  where all variables are in  $[-\delta, \delta]$ . For the other direction, note that because of the way in which the  $\text{LargerZero}$  constraint was defined, any solution of  $\Psi$  also corresponds to a solution of  $\Phi$ . So this proves that  $\Phi$  and  $\Psi$  are equisatisfiable and that all solutions of  $\Psi$  satisfy the necessary conditions.

Note that the number of variables  $m$  in the new CCI-EXPL instance depends in a linear manner on the number of variables  $n$  in  $\Phi$ . If we also want to ensure that  $\delta = T(m)$ , then we can define  $T^*(n)$  to be a bounded nicely computable function such that  $T(m) = T^*(n)$  for every possible value of  $n$ . Then we can decide to only consider CCI-EXPL\* instances with  $\delta = T^*(n)$  in the described construction. ■

In the next lemma, we allow even more possible  $f$  and  $g$ .

**LEMMA 3.11.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions such that  $f(x) = ax + bx^2 + O(x^3)$  and  $g(x) = cx + dx^2 + O(x^3)$  as  $x \rightarrow 0$ , where  $a, b, c, d \in \mathbb{Q}$  and  $b, d > 0$ . Let  $T$  be bounded and nicely computable. In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when only considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** We define  $f^*$  and  $g^*$  as  $f^*(x) = (f(x) - ax)/b$  and  $g^*(x) = (g(x) - cx)/d$ . From the constraints on  $f$  and  $g$  it follows that  $f^*(x) = x^2 + O(x^3)$  and  $g^*(x) = x^2 + O(x^3)$ . Therefore we can apply the previous lemma to these functions to find that the CCI-EXPL problem with  $f^*$  and  $g^*$  is  $\exists\mathbb{R}$ -hard. We will denote this problem by CCI-EXPL\*, and give a reduction from CCI-EXPL\* to the CCI-EXPL problem with  $f$  and  $g$  as defined in the lemma statement.

Let  $(\delta, \Phi)$  be any instance of CCI-EXPL\*. We denote

$$\delta' = (1 + |a| + |b| + |c| + |d|)\delta.$$

Now we build an instance  $(\delta', \Psi)$  of CCI-EXPL in the following manner: We start by adding a variable  $\llbracket \delta \rrbracket$  which is meant as a replacement for the  $\delta$  in conditions of the form  $x = \delta$  in  $\Phi$ . To introduce this variable, we introduce an auxiliary variable  $\llbracket \delta' \rrbracket$  and enforce the following constraints:

$$\begin{aligned} \llbracket \delta' \rrbracket &= \delta', \\ \llbracket \delta' \rrbracket &= (1 + |a| + |b| + |c| + |d|) \llbracket \delta \rrbracket. \end{aligned}$$

We also add every variable of  $\Phi$  and all constraints of the form  $x + y = z$  or  $x \geq 0$  from  $\Phi$  to  $\Psi$ , and for every constraint  $x = \delta$  in  $\Phi$  we add a constraint  $x = \llbracket \delta \rrbracket$  to  $\Psi$ .

For every constraint  $y \geq f^*(x)$  of  $\Phi$  we introduce a new variable  $\llbracket ax + by \rrbracket$  to  $\Psi$  which we force to equal  $ax + by$  using some linear constraints. Furthermore we add a constraint  $\llbracket ax + by \rrbracket \geq f(x)$ . For constraints of the form  $y \leq g^*(x)$  we do something similar.

In this way,  $(\delta', \Psi)$  is a valid CCI-EXPL instance since all values of the variables in any solution can be seen to be bounded by  $\delta'$  using the triangle inequality. Furthermore, the new instance  $\Psi$  differs from  $\Phi$  only by new auxiliary variables and otherwise has exactly the same constraints on the original variables. Thus  $V(\Psi)$  is non-empty if and only if  $V(\Phi)$  is non-empty, and  $\Psi$  is domain adherent since  $\Phi$  is.

Finally, we will show that we might impose  $\delta' = T(n)$  on the instances. Note that the number of variables created by the reduction is linear in the number of old variables and the

number of constraints of the form  $y \geq f^*(x)$  or  $y \leq g^*(x)$  in  $\Phi$ . If  $\Phi$  has  $n$  variables, then there can be at most  $O(n^2)$  different constraints of one of these two forms, so we can find some integer constant  $k$  such that  $\Psi$  has at most  $k \cdot n^2$  variables. Now we can adjust the previous reduction to add sufficiently many extra variables (not occurring in any constraint) to make sure there are exactly  $k \cdot n^2$  variables. We can also define  $T^*$  as  $T^*(n) = \frac{T(kn^2)}{1+|a|+|b|+|c|+|d|}$ , now restricting the inputs of CCI-EXPL\* to cases with  $\delta = T^*(n)$  gives the desired result. ■

The next step is to notice that any function which is three times differentiable with a nonzero second derivative satisfies the constraints from the previous lemma. This leads to the following result:

**COROLLARY 3.7.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f, g: U \rightarrow \mathbb{R}$  be functions which are three times differentiable such that  $f(0) = g(0) = 0$  and  $f'(0), f''(0), g'(0), g''(0) \in \mathbb{Q}$  with  $f''(0), g''(0) > 0$ . Let  $T$  be bounded and nicely computable. In this setting, CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** Using Taylor's theorem, we find that

$$f(x) = f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3) \text{ and}$$

$$g(x) = g'(0)x + \frac{g''(0)}{2}x^2 + O(x^3).$$

To this, we can apply the previous lemma to find that CCI-EXPL is  $\exists\mathbb{R}$ -hard, even when we only consider instances with  $\delta = T(n)$ . ■

### 3.4 Implicit Constraints

Using Corollary 3.7 we will show Theorem 1.13 and Theorem 1.11 in this order. Lemma 3.13 is almost equivalent to Theorem 1.13. The only difference is that the conditions  $f_y(0, 0) > 0$  and  $g_y(0, 0) < 0$  are added. As a preparation, we need the Implicit function theorem. We state the exact version that we use here for the convenience of the reader.

**THEOREM 3.12 (Implicit Function Theorem).** *Let  $U \subseteq \mathbb{R}^2$  be a neighborhood of  $(0, 0)$ . Let  $f: U \rightarrow \mathbb{R}$  be a  $C^3$ -function with  $f_y(0, 0) \neq 0$  defining the set  $S = \{(x, y) \in U \mid f(x, y) = 0\}$ . Now there is a neighborhood  $U' \subseteq \mathbb{R}^2$  of 0 with  $(U')^2 \subseteq U$  and some  $C^3$ -function  $f_{\text{expl}}: U' \rightarrow \mathbb{R}$  such that  $\{(x, y) \in (U')^2 \mid y = f_{\text{expl}}(x)\} = S \cap (U')^2$ . Furthermore, we have  $f'_{\text{expl}}(x) = -\frac{f_x(x, f_{\text{expl}}(x))}{f_y(x, f_{\text{expl}}(x))}$  for all  $x \in U'$ .*

**LEMMA 3.13.** *Let  $U \subseteq \mathbb{R}^2$  be a neighborhood of the origin. Let  $f, g: U \rightarrow \mathbb{R}$  be two functions, with  $f$  well-behaved and convexly curved, and  $g$  well-behaved and concavely curved. Furthermore assume that their partial derivatives satisfy  $f_y(0, 0) > 0$  and  $g_y(0, 0) < 0$ . Let  $T$  be bounded and nicely computable. Then the problem CCI is  $\exists\mathbb{R}$ -hard, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** Using the implicit function theorem, we find that in a neighborhood  $(U')^2 \subseteq U$  of  $(0, 0)$ , the curve  $f(x, y) = 0$  can also be given in an explicit form  $y = f_{\text{expl}}(x)$ , where  $f_{\text{expl}}$  is some  $C^3$ -function  $U' \rightarrow \mathbb{R}$ . So for  $(x, y) \in (U')^2$  we have  $f(x, y) = 0$  if and only if  $y = f_{\text{expl}}(x)$ . Since  $f_y(0, 0) > 0$ , it also follows that  $f(x, y) \geq 0$  if and only if  $y \geq f_{\text{expl}}(x)$ . Furthermore, the implicit function theorem also states that the derivative of  $f_{\text{expl}}$  is given by

$$f'_{\text{expl}}(x) = -\frac{f_x(x, f_{\text{expl}}(x))}{f_y(x, f_{\text{expl}}(x))}.$$

From this, it can be computed that the second derivative in 0 is

$$f''_{\text{expl}}(0) = -\left(\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{f_y^3}\right)(0, 0).$$

Note that the fact that  $f$  is convexly curved exactly implies that the numerator of this expression is a positive number. Using the assumptions from the lemma statement, we conclude that  $f''_{\text{expl}}(0)$  is a positive rational number.

In an analogous way, we can write the condition  $g(x, y) \geq 0$  in the form  $y \leq g_{\text{expl}}(x)$  in some neighborhood of  $(0, 0)$ , where  $g_{\text{expl}}$  is a  $C^3$ -function with rational first and second derivative in 0, and with positive second derivative. Without loss of generality we assume that  $f_{\text{expl}}$  and  $g_{\text{expl}}$  have the same domain  $U'$ .

We would now like to conclude that the problem CCI is equivalent to the problem CCI-EXPL for  $f_{\text{expl}}$  and  $g_{\text{expl}}$ , but we need to be slightly careful because of the exact way in which the constraints involving  $f$  and  $g$  are defined. Recall that with the constraint  $f(x, y) \geq 0$  in an CCI instance we actually mean  $\text{LargerZero}(f)$ , which was defined to be satisfied whenever  $(x, y)$  falls outside of  $U$ . Similarly the constraint  $y \geq f_{\text{expl}}(x)$  in an CCI-EXPL instance is satisfied whenever  $x \notin U'$ .

We will give a reduction from CCI-EXPL to CCI. Let  $\delta_0$  be some constant such that  $[-\delta_0, \delta_0] \subseteq U'$ . Let  $0 < c \leq \frac{1}{2}$  be a rational constant such that  $cT(n) \leq \delta_0$  for all  $n$ . This exists, since  $T$  is bounded. Now let  $(\Phi, \delta')$  be a CCI-EXPL instance with  $\delta' = T^*(n)$ . Here  $T^*$  will be determined later in such a way that  $T^*(n) = cT(m)$ , where  $m$  is the number of variables in the CCI instance  $(\Psi, \delta)$  which we will construct now. We start constructing  $\Psi$  by adding variables  $[\delta']$  and  $[\delta]$ , with linear constraints enforcing  $[\delta] = \delta$  and  $[\delta'] = c[\delta]$ . Next we add all linear constraints from  $\Phi$  to  $\Psi$ , except that we replace constraints of the form  $x = \delta'$  by  $x = [\delta']$ . For every constraint  $y \geq f_{\text{expl}}(x)$  in  $\Phi$ , we add constraints to  $\Psi$  enforcing  $f(x, y) \geq 0$ ,  $y \geq -\delta'$  and  $y \leq \delta'$ . Similarly, we replace  $y \leq g_{\text{expl}}(x)$  by constraints enforcing  $g(x, y) \geq 0$ ,  $y \geq -\delta'$  and  $y \leq \delta'$ .

Since  $\delta' \leq \delta_0$ , it follows that the constraints  $\text{LargerZero}(y - f_{\text{expl}}(x))$  and  $\text{LargerZero}(f)$  exactly coincide when restricted to  $[-\delta', \delta']^2$ . Since all solutions to  $\Phi$  are promised to have all coordinates in  $[-\delta', \delta']$ , it follows that every solution to  $\Phi$  gives rise to a solution of  $\Psi$ . From the definition of the  $\text{LargerZero}$  constraint it also follows that  $\text{LargerZero}(f)$  is satisfied whenever both  $\text{LargerZero}(y - f_{\text{expl}}(x))$  and  $y \in U$ . From this it follows that also every solution to  $\Psi$



corresponds to a solution of  $\Phi$ . Since all solutions of  $\Psi$  correspond to solutions of  $\Phi$ , it also follows that all solutions of  $\Psi$  are contained in  $[-\delta, \delta]^m$  and are domain adherent.

To make the correct choice of  $T^*$ , we proceed as in the end of the proof of Lemma 3.11. That is, we add some extra variables to  $\Psi$  to ensure that the total number of variables is always exactly  $m = kn^2$  for some constant  $k$ , and choose  $T^*(n) = cT(kn^2)$  for all  $n$ . ■

From here it is a small step to prove the main result:

**THEOREM 1.13. (Restated)** *Let  $f, g: U \rightarrow \mathbb{R}$  be well-behaved and triple algebraic. Furthermore, let  $f, g$  be respectively convexly curved and concavely curved. Let  $T$  be bounded and nicely computable. In this setting, CCI is  $\exists\mathbb{R}$ -complete, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** Without loss of generality, we may assume that  $f_y(0, 0) \neq 0$  and  $g_y(0, 0) \neq 0$ . In any other case, we can just interchange the variables in one of the functions.

In the case where  $f_y(0, 0) > 0$  and  $g_y(0, 0) < 0$ , we can apply the previous lemma and we are done. For the case  $f_y(0, 0) < 0$  and  $g_y(0, 0) < 0$ , we can provide a reduction from CCI with functions  $f^*(x, y) = f(-x, -y)$  and  $g^*(x, y) = g(x, y)$ . For the case  $f_y(0, 0) > 0$  and  $g_y(0, 0) > 0$  we can make a reduction from CCI with functions  $f^*(x, y) = f(x, y)$  and  $g^*(x, y) = g(-x, -y)$ . and for the case  $f_y(0, 0) < 0$  and  $g_y(0, 0) > 0$ , we can provide a reduction from CCI with functions  $f^*(x, y) = f(-x, -y)$  and  $g^*(x, y) = g(-x, -y)$ . Note that flipping the signs of the inputs of  $f$  or  $g$  does not influence any second partial derivative, while it does negate the first partial derivatives; therefore the mentioned starting points for the reductions can all be seen to satisfy the conditions from Lemma 3.13.

As an example we discuss the case  $f_y(0, 0) > 0$  and  $g_y(0, 0) > 0$ . We want to give reduction from the problem CCI with functions  $f^*(x, y) = f(x, y)$  and  $g^*(x, y) = g(-x, -y)$ ; we denote this CCI variation by CCI\*. So suppose that  $(\delta, \Phi)$  is a CCI\* instance. Now we will construct a CCI instance  $(\delta, \Psi)$  (with the  $f$  and  $g$  from the theorem statement). We add every variable of  $\Phi$  to  $\Psi$ , and for every such variable  $x$  we also add an extra variable  $\llbracket -x \rrbracket$ , together with a constraint enforcing  $x + \llbracket -x \rrbracket = 0$ . Furthermore, we copy every constraint from  $\Phi$  to  $\Psi$ , except for constraints of the form  $g^*(x, y) \geq 0$ . These constraints are replaced instead by  $g(\llbracket -x \rrbracket, \llbracket -y \rrbracket) \geq 0$ . This finishes the construction.

If we also want to enforce that  $\delta = T(m)$  with  $m$  the number of variables in the new CCI instance, then we can define  $T^*$  as  $T^*(n) = T(2n)$  and only consider CCI\* instances with  $\delta = T^*(n)$ . Note that the reduction always doubles the number of variables, and therefore this implies that also  $\delta = T(m)$  with  $m$  the number of variables in  $\Psi$ . ■

As a final result in this section, we prove Theorem 1.11 as well. To do this, we start from Corollary 3.7 and convert this to a result about CE-EXPL.

**LEMMA 3.14.** *Let  $U \subseteq \mathbb{R}$  be a neighborhood of 0, and let  $f: U \rightarrow \mathbb{R}$  be a function which is three times differentiable such that  $f(0) = 0$  and  $f'(0), f''(0) \in \mathbb{Q}$  with  $f''(0) \neq 0$ . Let  $T$  be bounded and nicely computable. In this setting, CE-EXPL is  $\exists\mathbb{R}$ -hard, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** We apply Corollary 3.7 to the case where  $g = f$  to find that in this case CCI-EXPL is  $\exists\mathbb{R}$ -hard. We can reduce this problem to CE-EXPL. Let  $(\delta, \Phi)$  be a CCI-EXPL instance. Now we construct an CE-EXPL formula  $\Psi$ . We copy all constraints of the form  $x + y = z$ ,  $x \geq 0$  and  $x = \delta$  from  $\Phi$ . For every constraint  $y \geq f(x)$  we introduce two new variables  $\llbracket f(x) \rrbracket$  and  $\llbracket y - f(x) \rrbracket$ , which we restrict by constraints

$$\begin{aligned}\llbracket f(x) \rrbracket &= f(x), \\ y &= \llbracket y - f(x) \rrbracket + \llbracket f(x) \rrbracket, \text{ and} \\ \llbracket y - f(x) \rrbracket &\geq 0.\end{aligned}$$

In a similar manner we replace every constraint  $y \leq f(x)$  by introducing new variables  $\llbracket f(x) \rrbracket$  and  $\llbracket f(x) - y \rrbracket$  and imposing the constraints

$$\begin{aligned}\llbracket f(x) \rrbracket &= f(x), \\ \llbracket f(x) \rrbracket &= \llbracket f(x) - y \rrbracket + y, \text{ and} \\ \llbracket f(x) - y \rrbracket &\geq 0.\end{aligned}$$

This completes the construction. It can easily be checked that every solution of  $\Phi$  corresponds to a solution of  $\Psi$ , and vice versa.

In order to enforce that  $\delta = T(m)$  with  $m$  the number of variables  $\Psi$ , we use a technique similar to that used in Lemma 3.11. Again we add extra variables to ensure that the number of variables in  $\Psi$  is exactly  $kn^2$  for some constant  $k$ , where  $n$  is the number of variables in  $\Phi$ . Then we take  $T^*(n) = T(kn^2)$  and only consider instances  $(\delta, \Phi)$  of CCI-EXPL which satisfy  $\delta = T^*(n)$ . ■

**THEOREM 1.11. (Restated)** *Let  $f: U \rightarrow \mathbb{R}$  be a function that is well-behaved, curved, and triple algebraic. Let  $T$  be a function that is both bounded and nicely computable. In this setting, CE is  $\exists\mathbb{R}$ -complete, even when considering only instances where  $\delta = T(n)$ , with  $n$  being the number of variables.*

**PROOF.** This proof is very similar to that of Lemma 3.13. Without loss of generality, we may assume that  $f_y(0, 0) \neq 0$ , otherwise we can swap the variables. Using the implicit function theorem, we can write the condition  $f(x, y) = 0$  in some neighborhood  $(U')^2 \subseteq U$  of  $(0, 0)$  as  $y = f_{\text{expl}}(x)$ , where  $f_{\text{expl}}$  is some  $C^3$ -function  $U' \rightarrow \mathbb{R}$ . Using the fact that the curvature of  $f$  is nonzero, the implicit function theorem also tells us that  $f''_{\text{expl}}(0) \neq 0$ .

Now we give a reduction from CE-EXPL to CE. Choose  $\delta_0$  such that  $[-\delta_0, \delta_0] \subseteq U'$ , and let  $c \leq \frac{1}{2}$  be a positive rational number such that  $cT(n) \leq \delta_0$  for all  $n$ . Let  $(\Phi, \delta')$  be a CE-EXPL instance with  $\delta' = T^*(n)$  to be determined later. We will build an instance  $(\Psi, \delta)$ , where  $\delta = T(m)$  with  $m$  the final number of variables in  $\Psi$ . First we add variables  $[\delta], [\delta']$  to  $\Psi$  with constraints enforcing  $[\delta'] = c[\delta]$ . We replace all constraints  $x = \delta'$  in  $\Phi$  by  $x = [\delta']$ , and we transfer all other linear equalities and inequalities from  $\Phi$  to  $\Psi$  as well. Finally, we replace every constraint in  $\Phi$  of the form  $y = f_{\text{expl}}(x)$  by constraints enforcing  $y = f(x)$ ,  $y \geq -\delta'$  and  $y \leq \delta'$ .

For the same reasons as in the proof of Lemma 3.13, the solutions of  $\Phi$  are in correspondence with those of  $\Psi$ . Furthermore, the solutions of  $\Psi$  will again all be domain adherent and contained in  $[-\delta, \delta]^m$  with  $m$  the number of variables in  $\Psi$ . Finally, we may add some extra variables to  $\Psi$  to ensure it has exactly  $kn^2$  variables for some constant  $k$ , and then choose  $T^*(n) = cT(kn^2)$  for all  $n$ . ■

## References

- [1] Mikkel Abrahamsen. Covering polygons is even harder. *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 375–386. IEEE, 2022. DOI (6)
- [2] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The art gallery problem is  $\exists\mathbb{R}$ -complete. *J. ACM*, 69(1):4:1–4:70, 2022. Also in STOC 2018, 65–73. DOI (2, 6, 9, 12)
- [3] Mikkel Abrahamsen, Linda Kleist, and Tillmann Miltzow. Training neural networks is ER-complete. *Advances in Neural Information Processing Systems (NeurIPS 2021)*, pages 18293–18306, 2021. Also in arXiv:2102.09798. URL (6)
- [4] Mikkel Abrahamsen and Tillmann Miltzow. Dynamic toolbox for ETRINV. *arXiv:1912.08674*, 2019. DOI (1, 8, 9, 21, 23)
- [5] Mikkel Abrahamsen, Tillmann Miltzow, and Nadja Seiferth. Framework for  $\exists\mathbb{R}$ -completeness of two-dimensional packing problems. *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*, pages 1014–1021. IEEE, 2020. arXiv:2004.07558. DOI (2, 3, 12, 17, 21)
- [6] Eric Allender, Peter Bürgisser, Johan Kjeldgaard-Pedersen, and Peter Bro Miltersen. On the complexity of numerical analysis. *SIAM Journal on Computing*, 38(5):1987–2006, 2009. DOI (14)
- [7] Saugata Basu, Richard Pollack, and Marie-Françoise Roy. *Algorithms in real algebraic geometry*. Springer, Berlin Heidelberg, 2006. DOI (6)
- [8] Saugata Basu and Marie-Françoise Roy. Bounding the radii of balls meeting every connected component of semi-algebraic sets. *Journal of Symbolic Computation*, 45(12):1270–1279, 2010. MEGA2009. DOI (20, 25, 27)
- [9] Marie Louisa Tølbøll Berthelsen and Kristoffer Arnsfelt Hansen. On the computational complexity of decision problems about multi-player nash equilibria. *Theory of Computing Systems*, 66(3):519–545, 2022. DOI (6)
- [10] Daniel Bertschinger, Christoph Hertrich, Paul Jungeblut, Tillmann Miltzow, and Simon Weber. Training fully connected neural networks is  $\exists\mathbb{R}$ -complete. *Advances in Neural Information Processing Systems (NeurIPS 2023)*, volume 36, pages 36222–36237. Curran Associates, Inc., 2023. URL (6)
- [11] Vittorio Bilò and Marios Mavronicolas.  $\exists\mathbb{R}$ -complete decision problems about (symmetric) Nash equilibria in (symmetric) multi-player games. *ACM Trans. Econ. Comput.* 9(3), 2021. DOI (6)
- [12] Vittorio Bilò and Marios Mavronicolas. A catalog of EXISTS-R-complete decision problems about Nash equilibria in multi-player games. *33rd Symposium on Theoretical Aspects of Computer Science (STACS 2016)*, volume 47 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 17:1–17:13, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2016. DOI (6)
- [13] Manuel Bodirsky and Martin Grohe. Non-dichotomies in constraint satisfaction complexity. *Automata, Languages and Programming (ICALP 2008)*, pages 184–196. Springer, 2008. DOI (4, 15)

- [14] Manuel Bodirsky, Peter Jonsson, and Timo Von Oertzen. Essential convexity and complexity of semi-algebraic constraints. *Logical Methods in Computer Science*, Volume 8, Issue 4, October 2012. DOI (4, 14)
- [15] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. *J. ACM*, 57(2):1–41, 2010. DOI (4)
- [16] Manuel Bodirsky and Marcello Mamino. Constraint satisfaction problems over numeric domains. Andrei Krokhin and Stanislav Zivny, editors, *The Constraint Satisfaction Problem: Complexity and Approximability*. Volume 7, Dagstuhl Follow-Ups, pages 79–111. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl, Germany, 2017. DOI (5, 14)
- [17] Stephen Boyd, Stephen P. Boyd, and Lieven Vandenbergh. *Convex Optimization*. Cambridge University Press, 2004. DOI (14)
- [18] Andrei A Bulatov. A dichotomy theorem for nonuniform CSPs. *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 319–330. IEEE, 2017. DOI (4)
- [19] Andrei A. Bulatov. A dichotomy theorem for constraint satisfaction problems on a 3-element set. *J. ACM*, 53(1):66–120, 2006. DOI (4)
- [20] John Canny. Some algebraic and geometric computations in PSPACE. *Proceedings of the Twentieth Annual ACM Symposium on Theory of Computing (STOC 1988)*, pages 460–467, New York, NY, USA. Association for Computing Machinery, 1988. DOI (5)
- [21] Jean Cardinal, Stefan Felsner, Tillmann Miltzow, Casey Tompkins, and Birgit Vogtenhuber. Intersection graphs of rays and grounded segments. *Journal of Graph Algorithms and Applications*, 22(2):273–295, 2018. Also in WG 2017. DOI (6)
- [22] Jean Cardinal and Udo Hoffmann. Recognition and complexity of point visibility graphs. *DCG*, 57(1):164–178, 2017. DOI (6)
- [23] Dmitry Chistikov, Stefan Kiefer, Ines Marusic, Mahsa Shirmohammadi, and James Worrell. On Restricted Nonnegative Matrix Factorization. *International Colloquium on Automata, Languages and Programming (ICALP 2016)*, volume 55 of *LIPICs*, 103:1–103:14, 2016. arXiv:1605.06848. DOI (6)
- [24] Argyrios Deligkas, John Fearnley, Themistoklis Melissourgos, and Paul G. Spirakis. Approximating the Existential Theory of the Reals. *Journal of Computer and System Sciences*, 125:106–128, 2022. DOI (29)
- [25] Michael G. Dobbins, Linda Kleist, Tillmann Miltzow, and Paweł Rzażewski.  $\forall\mathbb{R}$ -completeness and area-universality. *Graph-Theoretic Concepts in Computer Science (WG 2018)*, pages 164–175, Cham. Springer International Publishing, 2018. DOI (2, 6, 15)
- [26] Michael Gene Dobbins, Andreas Holmsen, and Tillmann Miltzow. A universality theorem for nested polytopes. *arXiv:1908.02213*, 2019. DOI (2, 6)
- [27] Martin Dyer, Leslie Ann Goldberg, and Mark Jerrum. An approximation trichotomy for Boolean #CSP. *Journal of Computer and System Sciences*, 76(3-4):267–277, 2010. DOI (4)
- [28] Jeff Erickson. Optimal curve straightening is  $\exists\mathbb{R}$ -complete. *arXiv:1908.09400*, 2019. DOI (6)
- [29] Jeff Erickson, Ivor van der Hoog, and Tillmann Miltzow. Smoothing the gap between NP and ER. *SIAM Journal on Computing*, 0(0):FOCS20-102-FOCS20-138, 2022. Also in FOCS 2020. DOI (4, 6)
- [30] Jugal Garg, Ruta Mehta, Vijay V. Vazirani, and Sadra Yazdanbod.  $\exists\mathbb{R}$ -completeness for decision versions of multi-player (symmetric) Nash equilibria. *ACM Trans. Econ. Comput.* 6(1), January 2018. DOI (6)
- [31] Thierry Gensane and Philippe Ryckelynck. Improved dense packings of congruent squares in a square. *DCG*, 34(1):97–109, 2005. DOI (7)
- [32] Simon Hengeveld and Tillmann Miltzow. A practical algorithm with performance guarantees for the art gallery problem. *37th International Symposium on Computational Geometry (SoCG 2021)*, volume 189 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 44:1–44:16, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021. DOI (7)
- [33] Gabriela Jeronimo and Daniel Perrucci. On the minimum of a positive polynomial over the standard simplex. *Journal of Symbolic Computation*, 45(4):434–442, 2010. DOI (30)
- [34] Peter Jonsson and Johan Thapper. Constraint satisfaction and semilinear expansions of addition over the rationals and the reals. *Journal of Computer and System Sciences*, 82(5):912–928, 2016. DOI (4)
- [35] Ross Kang and Tobias Müller. Sphere and dot product representations of graphs. *Proceedings of the Twenty-Seventh Annual Symposium on Computational Geometry (SoCG 2011)*, pages 308–314, New York, NY, USA. ACM, Association for Computing Machinery, 2011. DOI (6)
- [36] Fabian Klute and Tillmann Miltzow. Map labelling is  $\exists\mathbb{R}$ -complete. *in preparation*, 2022. (17)
- [37] Jan Kratochvíl and Jiří Matoušek. Intersection graphs of segments. *Journal of Combinatorial Theory, Series B*, 62(2):289–315, 1994. DOI (6)
- [38] Anna Lubiw, Tillmann Miltzow, and Debajyoti Mondal. The complexity of drawing a graph in a polygonal region. *Journal of Graph Algorithms and Applications*, 26(4):421–446, 2022. DOI (2, 6, 8, 15)
- [39] David Marker, Ya'acov Peterzil, and Anand Pillay. Additive reducts of real closed fields. *The Journal of Symbolic Logic*, 57(1):109–117, 1992. DOI (13)
- [40] Dániel Marx. Parameterized complexity of constraint satisfaction problems. *Computational Complexity*, 14(2):153–183, 2005. Also in CCC 2004. DOI (4)

- [41] Jiří Matoušek. Intersection graphs of segments and  $\exists\mathbb{R}$ . *arXiv:1406.2636*, 2014. DOI (5–7, 15)
- [42] Colin McDiarmid and Tobias Müller. Integer realizations of disk and segment graphs. *Journal of Combinatorial Theory, Series B*, 103(1):114–143, 2013. DOI (6)
- [43] Nicolai Mněv. The universality theorems on the classification problem of configuration varieties and convex polytopes varieties. *Topology and geometry – Rohlin seminar*, pages 527–543, 1988. DOI (6)
- [44] Ryan O’Donnell. SOS is not obviously automatizable, even approximately. *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, volume 67 of *Leibniz International Proceedings in Informatics (LIPIcs)*, 59:1–59:10, Dagstuhl, Germany. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017. DOI (14)
- [45] Gábor Pataki and Aleksandr Touzov. How do exponential size solutions arise in semidefinite programming? *SIAM Journal on Optimization*, 34(1):977–1005, 2024. DOI (14)
- [46] Ya’acov Peterzil. A structure theorem for semibounded sets in the reals. *The Journal of Symbolic Logic*, 57(3):779–794, 1992. DOI (13)
- [47] Ya’acov Peterzil. Reducts of some structures over the reals. *The Journal of Symbolic Logic*, 58(3):955–966, 1993. DOI (13)
- [48] Jürgen Richter-Gebert and Günter M. Ziegler. Realization spaces of 4-polytopes are universal. *Bulletin of the American Mathematical Society*, 32(4):403–412, 1995. DOI (6)
- [49] Marcus Schaefer. Complexity of geometric  $k$ -planarity for fixed  $k$ . *Journal of Graph Algorithms and Applications*, 25(1):29–41, 2021. DOI (6)
- [50] Marcus Schaefer. Complexity of some geometric and topological problems. *GD 2009, Lecture Notes in Computer Science (LNCS)*, pages 334–344. Springer, 2009. DOI (6)
- [51] Marcus Schaefer. Realizability of graphs and linkages. János Pach, editor, *Thirty Essays on Geometric Graph Theory*, chapter 23, pages 461–482. 2013. DOI (6, 27)
- [52] Marcus Schaefer and Daniel Štefankovič. Fixed points, Nash equilibria, and the existential theory of the reals. *Theory of Computing Systems*, 60(2):172–193, 2017. DOI (6, 23, 25, 30)
- [53] Marcus Schaefer and Daniel Štefankovič. The complexity of tensor rank. *Theory of Computing Systems*, 62(5):1161–1174, 2018. DOI (6)
- [54] Thomas J. Schaefer. The complexity of satisfiability problems. *Proceedings of the Tenth Annual ACM Symposium on Theory of Computing (STOC 1978)*, pages 216–226, New York, NY, USA. Association for Computing Machinery, 1978. DOI (4)
- [55] Yaroslav Shitov. A universality theorem for nonnegative matrix factorizations. *arXiv:1606.09068*, 2016. DOI (6)
- [56] Yaroslav Shitov. The complexity of positive semidefinite matrix factorization. *SIAM Journal on Optimization*, 27(3):1898–1909, 2017. DOI (6)
- [57] Peter Shor. Stretchability of pseudolines is NP-hard, *Applied Geometry and Discrete Mathematics*, pages 531–554. 1991. DOI (6)
- [58] Jack Stade. The point-boundary art gallery problem is  $\exists\mathbb{R}$ -hard. *CoRR*, abs/2210.12817, 2022. DOI (6)
- [59] Sergey P. Tarasov and Mikhail N. Vyalyi. Semidefinite programming and arithmetic circuit evaluation. *Discrete Applied Mathematics*, 156(11):2070–2078, 2008. In Memory of Leonid Khachiyan (1952 - 2005 ). DOI (14)
- [60] Caterina Viola and Stanislav Živný. The combined basic LP and affine IP relaxation for promise VCSPs on infinite domains. *ACM Trans. Algorithms*, 17(3), July 2021. DOI (4)
- [61] Wikipedia. Galois theory — Wikipedia, the free encyclopedia, 2023. [Online; accessed 7-July-2023]. URL (19)
- [62] Xiao-Dong Zhang. Complexity of neural network learning in the real number model. *Workshop on Physics and Computation*, pages 146–150, 1992. DOI (6)
- [63] Dmitriy Zhuk. A Proof of the CSP Dichotomy Conjecture. *J. ACM*, 67(5):1–78, 2020. DOI (4)
- [64] Dmitriy Zhuk, Barnaby Martin, and Michał Wrona. The complete classification for quantified equality constraints. *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2746–2760. DOI (4)

## A. Circle-Constraint

Here, we discuss the question of expressing multiplication via linear equations and the circle constraint  $x^2 + y^2 = 1$ . We note that for real numbers  $x$  and  $y$  the following equivalence holds:

There exists a real  $z$  such that  $z^2 + (x + y)^2 = 1$  and  $(z + x - y)^2 + (z - x + y)^2 = 1$  if and only if  $8xy = 1$  and  $|x + y| \leq 1$ . This can be used in turn to express the inversion constraint

$(x \cdot y = 1)$  after some scaling and imposing range constraints. Note that  $x \cdot y = 1$  can be used to express squaring as follows

$$\frac{1}{\frac{1}{x} - \frac{1}{x+1}} - x = x^2.$$

And we saw already in the Section 1.2 how squaring can be used to express multiplication.